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INTERNATIONAL MATHEMATICS SUMMER CAMP-IMSC23 COMPETITION

Shortlisted Problems (Solutions)

CONFIDENTIAL

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3. Problems-Algebra

Problem A1. Let $x, y, z > 0$ such that $xyz = 1$. Show that

$$\frac{1}{x^3(y+z)^2} + \frac{1}{y^3(z+x)^2} + \frac{1}{z^3(x+y)^2} \geq \frac{3}{4} \cdot \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{x+y+z}.$$

Problem A2. We call a given 2022-tuple of nonnegative reals $(l_1, l_2, \ldots, l_{2022})$ *campy* if

$$2022 \sum_{i=1}^{2022} l_i^2 = 2023 + 2 \sum_{1 \leqslant i < j \leqslant 2022} l_i l_j.$$

Find

$$\min_{(l_1, l_2, \ldots, l_{2022})} (\max\{l_1, l_2, \ldots, l_{2022}\}),$$

where the minimum is taken over all campy 2022-tuples.

Problem A3. Let $x, y, z > 0$ such that $x + y + z = 1$. Prove that

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{x+z}} + \frac{1}{\sqrt{y+z}} \geq \frac{1}{\sqrt{x^4+y^4+z^4+xyz}}$$

Problem A4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(1) \neq f(-1)$ and

$$f(m+n)^2 \mid f(m) - f(n) \quad \text{for all } m, n \in \mathbb{Z}.$$

Problem A5. Let $n \geq 2$ and $t_1, t_2, \ldots, t_n$ be real numbers with $t_i \in [0, 1]$, for all $i \in \overline{1, n}$. Prove that

$$\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \leq \sum_{1 \leq i < j \leq n} \left(t_i t_j + \sqrt{(1-t_i^2)(1-t_j^2)} \right) \leq \frac{(n-1)n}{2}.$$

Problem A6. Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$ we have

$$n^3 - n^2 \leq f(n) \cdot (f(f(n)))^2 \leq n^3 + 2n^2.$$

Problem A7. Let $n > 1$ be an integer. Determine the smallest positive real m such that, for every n reals $x_1, x_2, \ldots, x_n \in [0, 1]$, there is a permutation $(y_1, y_2, \ldots, y_n)$ of $(x_1, x_2, \ldots, x_n)$, such that, letting $y_{n+1} = y_1$,

$$\sum_{k=1}^{n} \sqrt{|y_{k+1} - y_k|} \leq m$$

Problem A8. Initially, there is a non constant polynomial $P(x)$ with complex coefficients with nonzero constant term on the board. At each step, we can choose a complex number C and write $P(Cx)$ are $CP(x)$ on the board. If two (not necessarily distinct) polynomials $P(x), Q(x)$ are on the board, then we can write their composition or product on the board.

i. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of complex numbers, we call a polynomial $f(x)$ with complex coefficients nice if $f(A) = B$. Find all polynomials $P(x)$ that can be written on the board initially that for every two sets A, B, and such that after some of the above operations we can produce a nice polynomial.

ii. Let $P(x)$ has real coefficients and sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of real numbers. Find all polynomials $P(x)$ that can be written on the board initially that for every two sets A, B, and such that after some of the above operations we can produce a nice polynomial.

4. Problems-Combinatorics

Problem C1. Let a *brick* be a unit cube such that its three faces sharing a common vertex are colored blue, its two faces are colored yellow, and the remaining face is colored green. Find the largest positive integer n such that n^3 bricks (not necessarily identical ones) can be stacked to form a cube of side n, where for every two neighboring bricks their joining faces are either colored green in both bricks, or colored blue in one brick and yellow in the other brick.

Problem C2. A mathematician born in the 20^{th} century noted:

> My phone number has six digits. The three-digit number formed by the first three digits is different than the three-digit number formed by the last three digits. The first three digits are nonzero and mutually different, and if we consider all the other five three-digit integers that can be obtained by their permutation, the sum of them five gives my birth year. The last three digits are also nonzero and mutually different, and if we consider all the other five three-digit integers that can be obtained by their permutation, the sum of them five gives the birth year of my twin brother.

Was this mathematician born during a weekend?

Problem C3. Is it possible for a convex polygon to be partitioned into finitely many smaller convex polygons such that each of these smaller polygons has at least 6 sides and each vertex of these smaller polygons is also a vertex of at least two other smaller polygons?

Problem C4. A *binoku* is a 9×9 grid of zeros and ones such that each row and each column and each of the nine 3×3 subgrids contain at least one 0 and at least one 1. An *incomplete binoku* is a 9×9 grid containing zeros, ones, and empty cells. What is the largest number of empty cells that an incomplete binoku can contain if it can be completed into a binoku in a unique way?

Problem C5. In a plane, there are 2022 points are colored either black or white, in such a way that no three points lie on the same line, and that the triangle formed by every three black points contains at least one white point. What is the largest possible number of black points?

Problem C6. In a network of 60 metro stations $1, 2, \ldots, 60$, there are direct connections $C_{i,j}$ between some stations $i < j$. On such a connection, one can travel in either direction for one pound. Let a_i be the number of stations one can travel to for one pound starting from station i (station i not included). For any connection $C_{i,j}$, let $L_{i,j}$ be the number of stations that are (strictly) cheaper to travel to from station i than from station j. Note that station i is one such station. For any connection $C_{i,j}$, let $H_{i,j}$ be the number of stations that are more expensive to travel to from station i than from station j. Note that station j is one such station. What is the maximum possible value of

$$\sum_{C_{i,j}\in E} (a_i + a_j) \cdot L_{i,j} \cdot H_{i,j}$$

where E is the set of all connections?

Problem C7. There are $n!$ baskets in a row, numbered $1, 2 \ldots, n!$. John first puts a stone in every basket. John then puts 2 stones in every second basket. John repeats this until putting n stones into every nth basket. In other words, for each $i = 1, 2, \ldots, n$, John puts i stones into the baskets labeled $i, 2i, 3i, \ldots, n!$.

Let x_i be the number of stones in basket i after all these steps. Show that

$$n! \cdot n^2 \le \sum_{i=1}^{n!} x_i^2 \le n! \cdot n^2 \cdot \sum_{i=1}^{n} \frac{1}{i}.$$

Problem C8. A rectangular grid is divided by two perpendicular straight lines into four smaller rectangles with integral side lengths. It is possible to remove one among these four rectangles in such a way that the remaining figure can be exactly covered by rectangles of size 2×3 and 3×2. Prove that it is possible to exactly cover one among these four smaller rectangles by rectangles of size 2×3 and 3×2. (By *exact covering* we mean covering without gaps, overlaps and overflows.)

5. Problems-Geometry

Problem G1. Let $\mathbb{R}$ denote the set of the real numbers and let $\mathbb{R}^2$ denote the Euclidean plane. Find all functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that for every non-degenerate triangle ABC with orthocenter H, the following equation holds:

$$f(H) = \frac{1}{3}(f(A) + f(B) + f(C)).$$

Problem G2. Let ABC be a triangle with incentre I. Let D be the intersection of line AI and line BC. Let E be the point on segment AC such that $CD = CE$. Similarly, let F be the point on segment AB such that the length of BF is equal to the length of BD. Let P be the intersection, that is different from I, of the circumcircle of the triangle CEI and the circumcircle of the triangle DFI. Similarly, let Q be the intersection, that is different from I, of the circumcircle of the triangle BFI and the circumcircle of the triangle DEI. Prove that PQ is orthogonal to BC.

Problem G3. Let ABC be a triangle with circumcircle Γ. Let ω_A be the circle such that the center of ω_A is on BC and ω_A is tangent to both AB and AC. Suppose that Γ and ω_A intersect at two distinct points, denoted by X_A and Y_A. Let A' be the intersection of the line BC and the line X_AY_A. Define B' and C' analogously. Prove that A', B' and C' are collinear.

Problem G4. Let ABC be a triangle, and let its incircle touch the sides BC, CA, AB at points D, E, F, respectively. Let M and N be the midpoints of the segments DE and DF, respectively. Let P, that is different from B, be the intersection of the circumcircle of the triangle BDM and the circumcircle of the triangle ABC. Similarly, let Q, that is different from C, be the intersection of the circumcircle of the triangle CDN and the circumcircle of the triangle ABC. Prove that the points P, Q, M and N lie on a circle.

Problem G5. Suppose a circle centered at I is inscribed in a convex quadrilateral $ABCD$. Let E and F be respectively on BI and DI such that $\angle EAF = \frac{1}{2}\angle BAD$. Let X be the intersection of DE and BF, and let K be the intersection of IX and EF. Prove that EF bisects the angle AKC.

Problem G6. Let ABC be a triangle with circumcenter O and othocenter H. Let ℓ be the line through O and H. Let D be a point on the circumcircle of the triangle ABC such that the reflection of D across the line AB is a point on ℓ that is different from H. Let E be a point such that D is the midpoint of OE. Let F be the reflection of E across the bisector of $\angle ACB$. Let G be an arbitrary point on the line CF, and let K be the reflection of G across the bisector of $\angle ABC$. Let P, Q, R be the foots of perpendiculars from the point G to the lines BC, CA, AB respectively. Let X, Y, Z be the intersections of lines GA, GB, GC with the lines BC, CA, AB respectively. Suppose that $\angle PQR = \angle XYZ$. Prove that B, E and K are collinear.

6. Problems-Number Theory

Problem N1. Find all integer values of a for which $X^2 + X + a$ divides $X^{13} + X - 90$.

Problem N2. For $n > 1$, define the function $C(n)$ as follows:

$$C(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 2), \text{ where } n = \prod_{i=1}^{k} p_i^{\alpha_i}, p_i \text{ prime}$$

Prove that if

$$C(n) | \phi(n) - 1,$$

then n is an odd prime or n has at least 7 prime distinct divisors.

Problem N3. Determine all positive primes p such that there exists integers x and y such that x and y are not divisible by p and $x^2 + y^3 + 1$ is divisible by p.

Problem N4. Let a be an odd number. Define the sequence $\{x_n\}_{n \geq 0}$ by $x_0 = 1, x_1 = a$, and $x_{n+2} = 2ax_{n+1} - x_n$. Show that for every prime power p^l, there is a choice of sign for which

$$x_{p^l} \pm x_{p^{l-1}} \equiv 0 \pmod{p^l}.$$

Problem N5. Let n be a positive integer. Prove there exists an N such that, for every prime $p > N$ there exist n consecutive numbers such that all of them are quadratic residues.

Problem N6. For a positive integer n, the symbol $\underset{(n)}{!}$ is defined in the following way:

$$a\underset{(n)}{!} = \prod_{\substack{1 \leqslant i \leqslant a \\ i \equiv a \pmod{n}}} i.$$

Find all positive integers n, where $n \geqslant 2023$, which satisfy the following: there exist infinitely many positive integers k such that, for some positive integers $a_1, a_2, \ldots, a_k$ for which $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, $a_k - a_1 > n$, and which are all congruent modulo n, and some positive integer t, we have

$$a_1\underset{(n)}{!} \cdot a_2\underset{(n)}{!} \cdots a_k\underset{(n)}{!} = t^k.$$

Note. The problem can be proposed even without the constraint $n \geqslant 2023$, but this adds some casework to the solution.

Problem N7. Find all polynomials $P(x)$ with integer coefficients such that for all positive integers m, n we have:

$$m + n \mid P^{(m)}(n) - P^{(n)}(m).$$

7. Solutions-Algebra

Problem A1. Let $x, y, z > 0$ such that $xyz = 1$. Show that

$$\frac{1}{x^3(y+z)^2} + \frac{1}{y^3(z+x)^2} + \frac{1}{z^3(x+y)^2} \geq \frac{3}{4} \cdot \frac{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}{x+y+z}.$$

Proposed by *Cezar Lupu, P.R. China*

Solution 1. First, we show the following

Lemma. *Let $x, y, z > 0$ and $\alpha, \beta, \gamma > 0$. Then we have*

$$\frac{x^3}{\alpha^2} + \frac{y^3}{\beta^2} + \frac{z^3}{\gamma^2} \geq \frac{(x+y+z)^3}{(\alpha+\beta+\gamma)^2}.$$

The lemma is just a consequence of Hölder's inequality since we can write the inequality in the form

$$\left(\sum_{cyc} (a^{1/3})^3\right)\left(\sum_{cyc} (a^{1/3})^3\right)\left(\sum_{cyc}\left(\frac{x}{a^{2/3}}\right)^3\right) \geq \left(\sum_{cyc} x\right)^3.$$

Now, going back to our problem, by the previous lemma, we have

$$\sum_{cyc} \frac{(1/x)^3}{(y+z)^2} \geq \frac{\left(\sum \frac{1}{x}\right)^3}{4(\sum x)^2} = \frac{(\sum xy)^3}{4(\sum x)^2}.$$

We are left to show that

$$(xy + yz + zx)^2 \geq 3(x + y + z).$$

But this last inequality s a consequence of Newton's inequality, $xy + yz + zx \geq \sqrt{3xyz(x+y+z)}$ taking into account that $xyz = 1$. □

Solution 2. Clearing denominators and homogenizing, it remains to show

$$4\sum_{cyc}(x+y)^2(x+z)^2y^3z^3 \geq 3(xy+yz+zx)(x+y)^2(y+z)^2(z+x)^2xyz.$$

Expanding, we get that this is equivalent to

$$\sum_{sym} 4x^6y^5 + 8x^6y^4z + 4x^6y^3z^2 + 10x^5y^5z + 28x^5y^4z^2 + 14x^5y^3z^3 + 28x^4y^4z^3 \geq$$

$$3xyz\left(\sum_{sym} x^5y^3 + 3x^5y^2z + x^4y^4 + 9x^4y^3z + 7x^4y^2z^2 + 11x^3y^3z^2\right),$$

and collecting like terms, we get the equivalent inequality

$$\sum_{sym} 4x^6y^5 + 5x^6y^4z + 7x^5y^5z + x^5y^4z^2 \geq \sum_{sym} 5x^6y^3z^2 + 7x^5y^3z^3 + 5x^4y^4z^3,$$

which is true by Muirhead's inequality.

Problem A2. We call a given 2022-tuple of nonnegative reals $(l_1, l_2, \dots, l_{2022})$ *campy* if

$$2022 \sum_{i=1}^{2022} l_i^2 = 2023 + 2 \sum_{1 \leqslant i < j \leqslant 2022} l_i l_j.$$

Find

$$\min_{(l_1, l_2, \dots, l_{2022})} (\max\{l_1, l_2, \dots, l_{2022}\}),$$

where the minimum is taken over all campy 2022-tuples.

Proposed by *Bojan Basic, Armenia*

Solution. The answer: $\sqrt{\frac{2023}{1011 \cdot 1012}}$, where this value is achieved for the 2022-tuples of the form (up to permutation)

$$\Bigg(\underbrace{0, 0, \dots, 0}_{1011 \text{ or } 1010}, \underbrace{\sqrt{\frac{2023}{1011 \cdot 1012}}, \sqrt{\frac{2023}{1011 \cdot 1012}}, \dots, \sqrt{\frac{2023}{1011 \cdot 1012}}}_{1011 \text{ or } 1012 \text{ (respectively)}}\Bigg).$$

(*Remark.* This is quite simple to be guessed if one starts from the idea that in the optimal case all the nonzero numbers are equal; straightforward calculation then gives that their value will then be $\sqrt{\frac{2023}{1011 \cdot 1012}}$. For the rest of the proof it is necessary to have this construction in advance, that is, it will not naturally appear from the proof, which is why here it is explained how to guess it.)

The condition from the statement can be written as

$$2023 \left(\sum_{i=1}^{2022} l_i^2 - 1 \right) = \left(\sum_{i=1}^{2022} l_i \right)^2.$$

Let $g(l_1, l_2, \dots, l_{2022}) = \frac{\left(\sum_{i=1}^{2022} l_i\right)^2}{\sum_{i=1}^{2022} l_i^2 - 1}$. Note that in the interior of the unit ball centered at the origin the function g is negative, while by approaching the ball from outside it tends to ∞. Let B be the open ball centered at the origin and of radius $1 + \varepsilon$, where ε is chosen so that all the positive values of g inside that ball, as well as on the boundary, are larger than 2023. It is enough to show that the minimum of g over the set $\left[0, \sqrt{\frac{2023}{1011 \cdot 1012}}\right]^{2022} \setminus B$ equals 2023, and that it is reached only in some points from $\left\{0, \sqrt{\frac{2023}{1011 \cdot 1012}}\right\}^{2022}$ (let us note, as g is continuous and the considered set is closed, the minimum is indeed reached).

We have seen in the beginning that g reaches the value 2023. Suppose the contrary: the minimum is reached in some point $(L_1, L_2, \dots, L_{2022})$, where (w.l.o.g.) $0 < L_1 < \sqrt{\frac{2023}{1012 \cdot 1012}}$. By the choice of B, the minimum is not reached at its boundary. If we fix the values $l_2 = L_2, \dots, l_{2022} = L_{2022}$, we have that L_1 is the point in which a minimum of a univariate function of the form $\frac{(l_1 + C_1)^2}{l_1^2 + C_2}$ is reached over the open interval $\left(0, \sqrt{\frac{2023}{1011 \cdot 1012}}\right)$; however, since its first derivative equals $\frac{2(l_1+C_1)(l_1^2+C_2)-(l_1+C_1)^2\cdot 2l_1}{(l_1^2+C_2)^2}$, its sign is the same as the sign of the expression $C_2 - C_1l_1$, which means that the only stationary point is $l_1 = \frac{C_1}{C_2}$, and since $C_1 > 0$, there we have a maximum (and not minimum) of the considered function (if this point belongs to the considered interval at all); contradiction.

This proves the claim. Therefore, in order to finish the problem, it is enough to calculate the value of g at the points of the form $\left(0,\dots,0,\sqrt{\frac{2023}{1011\cdot 1012}},\dots,\sqrt{\frac{2023}{1011\cdot 1012}}\right)$. Assume that the expression $\sqrt{\frac{2023}{1011\cdot 1012}}$ appears t times (and zeros $2022-t$ times; we additionally have $t \geqslant 1$, because of the exclusion of B). Then g in that point has the value $\frac{(t\sqrt{\frac{2023}{1011\cdot 1012}})^2}{t\frac{2023}{1011\cdot 1012}-1}$, that is, $\frac{2023t^2}{2023t-1011\cdot 1012}$. The sign of the first derivative of this function (with respect to t) depends on the part $2023t - 2\cdot 1011\cdot 1012$, that is, the minimum is reached for $t = \frac{2\cdot 1011\cdot 1012}{2023}$, which is between 1011 and 1012. We note that in both the cases $t = 1011$ and $t = 1012$ we get the result 2023, which means that this is the required minimum, which was to be proved (and this indeed gives the 2022-tuples from the beginning). □

Problem A3. Let $x, y, z > 0$ such that $x + y + z = 1$. Prove that

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{x+z}} + \frac{1}{\sqrt{y+z}} \geq \frac{1}{\sqrt{x^4+y^4+z^4+xyz}}$$

Proposed by *Tudor Popescu, United States*

Solution. Using Hölder's inequality, we have that

$$\Big(\sum \frac{1}{\sqrt{x+y}}\Big)\Big(\sum \frac{1}{\sqrt{x+y}}\Big)\Big(\sum z^3(x+y)\Big) \geq (x+y+z)^3 = 1$$

Therefore, we have that

$$\Big(\sum \frac{1}{\sqrt{x+y}}\Big)^2 \geq \frac{1}{\sum z^3(x+y)} \Rightarrow \Big(\sum \frac{1}{\sqrt{x+y}}\Big) \geq \frac{1}{\sqrt{\sum z^3(x+y)}}$$

Note that Schur's inequality yields that

$$x^4+y^4+z^4+xyz(x+y+z) \geq \sum z^3(x+y)$$

hence

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{x+z}} + \frac{1}{\sqrt{y+z}} \geq \frac{1}{\sqrt{\sum z^3(x+y)}} \geq \frac{1}{\sqrt{x^4+y^4+z^4+xyz}}$$

as desired. □

Solution 2. Since the function $(1-x)^{-1/2}$ is convex, the left hand side is at least $3\sqrt{3/2}$ by Jensen's Inequality. It remains to show

$$x^4+y^4+z^4+xyz \geq \frac{2}{27}.$$

By Muirhead, we have $x^4+y^4+z^4 \geq \frac{1}{3}(x^3+y^3+z^3)$, so we need only to show that

$$x^3+y^3+z^3+3xyz \geq \frac{2}{9},$$

which is equivalent to

$$\sum_{sym} 7x^3 - 12x^2y + 5xyz \geq 0.$$

But this is evident by Schur's inequality,

$$\sum_{sym} x^3 - 2x^2y + xyz \geq 0,$$

as well as Muirhead's inequality.

Problem A4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(1) \neq f(-1)$ and

$$f(m+n)^2 \mid f(m) - f(n) \quad \text{for all } m, n \in \mathbb{Z}.$$

Proposed by *Liam Baker, South Africa*

Solution. Note that f is a solution if and only if $-f$ is. Since $f(1) \neq f(-1)$, we know that $f(0) \neq 0$ and by the above we can assume that $f(0) \geq 1$.

If $f(0) = 1$, then choosing $m = m, n = 0$ gives $f(m)^2 \mid f(m) - 1$. This implies that $f(m) \in \{-1, 1\}$. Every function for which $f(m) = \pm 1$ for every $m \in \mathbb{Z}$ and $f(1) \neq f(-1)$ satisfies the function equation.

If $f(0) = 2$, we have $f(m)^2 \mid f(m) - 2$ implying that $f(m) \in \pm\{1, 2\}$. Since $f(0)^2 \mid f(1) - f(-1)$, we have that one equals 2 and the other one -2. Since $4 \mid f(1)^2 \mid f(m+1) - f(-m)$ and $4 \mid f(-1)^2 \mid f(m) - f(-m-1)$, we can prove by induction on the absolute value of m that $f(m) = \pm 2$ for every $m \in \mathbb{Z}$.

If $f(0) \geq 3$, we have that $f(0)^2 \mid f(1) - f(-1)$ implies that one of $f(1), f(-1)$ is (strictly) larger than $f(0)$ in absolute value. But $f(1)^2 \mid f(1) - f(0)$ implies that this is not possible, since $0 \neq |f(1) - f(0)| < 2|f(1)| < f(1)^2$ in that case (and similar for $f(-1)$.

It is not hard to see that the two families of all solutions satisfy the equality. □

Problem A5. Let $n \geq 2$ and $t_1, t_2, \ldots, t_n$ be real numbers with $t_i \in [0, 1]$, for all $i \in \overline{1, n}$. Prove that

$$\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \leq \sum_{1 \leq i < j \leq n} \left(t_i t_j + \sqrt{(1-t_i^2)(1-t_j^2)} \right) \leq \frac{(n-1)n}{2}.$$

Proposed by *Irina-Roxana Popescu, Tudor Popescu, Eric Wang, United States*

Solution 1. **Claim.**

We first note that for $t_i, t_j \in [0, 1]$. $(1-t_i)(1-t_j) \leq \sqrt{(1-t_i^2)(1-t_j^2)} \leq 1 - t_i t_j$.

Proof of the claim. The inequality on the left is true by noting that $(1-t_i)^2 \leq 1 - t_i^2$.

The right one is after squaring equivalent with $t_i^2 + t_j^2 \geq 2t_i t_j$, which is true by AM-GM.

We now return to the initial one. Using the right estimate, we have that the sum is bouded by $\binom{n}{2}$.

Using the left estimate, we note that the sum is

$$\sum_{i,j}\left(t_it_j+(1-t_i)(1-t_j)\right).$$

Claim: the minimum of the expression is attained for a choice with all t_i being 0 or 1. For a fixed t_i, the sum is equal to a sum independent of t_i and $t_i\left(\sum_{j\neq i}t_j-\sum_{j\neq i}(1-t_j)\right)$. As such, the extremum is found when $t_i\in\{0,1\}$.

Finally, it is sufficient to assume that there are k terms equal to 1 and $n-k$ terms equal to 0. The expression equals $\binom{k}{2}+\binom{n-k}{2}\geq\binom{\lfloor\frac{n}{2}\rfloor}{2}+\binom{\lceil\frac{n}{2}\rceil}{2}$, which equals the left hand side.

Solution 2. The right hand side inequality is simple to prove using the arithmetic geometric inequality. This is because

$$\sum_{1\leq i<j\leq n}\left(t_it_j+\sqrt{(1-t_i^2)(1-t_j^2)}\right)\leq\sum_{1\leq i<j\leq n}\left(t_it_j+\frac{(1-t_i^2)+(1-t_j^2)}{2}\right),$$

which in turn is equal to

$$\sum_{1\leq i<j\leq n}\left(1-\frac{(t_i-t_j)^2}{2}\right)\leq\frac{(n-1)n}{2},$$

with equality if and only if $t_1=t_2=\ldots=t_n$.

For the other part, we'll use the fact that

$$\sqrt{(1-t_i^2)(1-t_j^2)}\geq\sqrt{(1-t_i)(1-t_j)}\geq(1-t_i)(1-t_j).$$

Therefore, we want to show that

$$\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\leq\sum_{1\leq i<j\leq n}\left(t_it_j+(1-t_i)(1-t_j)\right)=$$

$$=2\sum_{1\leq i\leq j\leq n}t_it_j-(n-1)\sum_{i=1}^{n}t_i+\frac{(n-1)n}{2}$$

Let $f_n:[0,1]^n\to\mathbb{R}$,

$$f_n=2\sum_{1\leq i\leq j\leq n}t_it_j-(n-1)\sum_{i=1}^{n}t_i+\frac{(n-1)n}{2}.$$

We have that f_n is continuous and it attains its minimum and maximum on $[0,1]^n$, which is compact. We want to find its absolute minimum.
For this, we first look at the critical points. By taking the derivative with respect to one variable and keeping the others constant, we obtain that

$$2(t_2+t_3+\ldots+t_n)-(n-1)=0$$

and the symmetric relations, hence $t_1=t_2=\ldots=t_n=\frac{1}{2}$.
The value of this function at the critical point $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ is $\frac{(n-1)n}{4}$. Then one can compute the values at the endpoints, which by symmetry we can assume to be points

of the form $(1,1,\dots,1,0,0,\dots,0)$, where there are l $1's$ and $n-l$ $0'$s. Therefore, we have that

$$f_n(1,1,\dots,1,0,0,\dots,0)=\frac{(l-1)l}{2}-(n-1)l+\frac{(n-1)n}{2},$$

which is a quadratic in l that has minimum $\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$. This can be shown by considering the two cases $n=2k-1$ and $n=2k$ and using the fact that remember that for a quadratic ax^2+bx+c with $a>0$ the absolute minimum is attained at $x=\frac{-b}{2a}$. ☐

Problem A6. Let $\mathbb{N}_0=\{0,1,2,3,\dots\}$. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that for all $n\in\mathbb{N}_0$ we have

$$n^3-n^2\le f(n)\cdot(f(f(n)))^2\le n^3+2n^2.$$

Proposed by *Alzubair Habibullah, Suadi Arabia*

Solution. Call a subset $G\subseteq\mathbb{N}_0$ a *good* set if it contains no consecutive integers (so the empty set is good). We will show that the only possible solutions are (1st Type)

$$f(n)=\begin{cases}n+1 & \text{if } n\in G\\ n-1 & \text{if } n-1\in G\\ n & \text{otherwise}\end{cases}$$

where G is an arbitrary good set, and (2nd Type)

$$f(n)=\begin{cases}0 & \text{if } n\in\{0,1\}\\ n+1 & \text{if } n\in G_2\\ n-1 & \text{if } n-1\in G_2\\ n & \text{otherwise}\end{cases}$$

where $G_2\subseteq\{2,3,4,...\}$ is an arbitrary good set.
Indeed, we can easily check that both solutions satisfy the given inequality (we can see this by noticing that $f(f(n))=n$ for every $n\ge 2$).
Suppose now f satisfies the given inequality. Notice that $f(n)=0$ implies $n^3-n^2\le 0$, which is $n\in\{0,1\}$.
Lemma$^{(*)}$: if $a,b\in\mathbb{N}_0$ with $b>\max(a,1)$, then $f(b)\ne f(a)$.
Proof. Notice that

$$f(b)\cdot\big(f(f(b))\big)^2\ge b^2(b-1)>(b+1)(b-1)^2\ge(a+2)a^2\ge f(a)\cdot\big(f(f(a))\big)^2$$

immediately gives $f(a)\ne f(b)$ and finishes the proof.
Consider now 2 cases, $Case1:\quad f(1)\ne f(0)$, then $(*)$ implies that f is injective. Let

$$B:=\{n\in\mathbb{N}_0\mid(f(0),...,f(n))\,\text{is a permutation of}\,(0,1,...,n)\}$$

and C be the complement of B. We prove the following statement by induction on $n \geq 0$:

$$P(n): \begin{cases} C_n = C \cap \{0,1,..,n\} \text{ is good, and} \\ f(n) = \begin{cases} n+1 & \text{if } n \in C \\ n-1 & \text{if } n-1 \in C \\ n & \text{otherwise} \end{cases} \end{cases}$$

Indeed, we start by plugging $n = 0$ we will get $f(0) \cdot \big(f(f(0))\big)^2 = 0$, which gives $f(f(0)) = 0$, so $f(0) \in \{0, 1\}$. If $f(0) = 1$ then $f(1) = f(f(0)) = 0$, and so $C_0 = C_1 = \{0\}$ are good, hence both $P(0), P(1)$ are satisfied with $1 \in B$. And if $f(0) = 0$ then $C_0 = \phi$ is good and so $P(0)$ is satisfied with $0 \in B$. (Notice that we either prove $P(n)$ with $n \in B$, or prove $P(n), P(n+1)$ together with $n+1 \in B$)
Suppose that $P(0), P(1), ..., P(n-1)$ are all satisfied for some $n \geq 1$ with $n-1 \in B$, then by injectivity we will have $f(m) \geq n$ whenever $m \geq n$, and so we'll have

$$n \cdot \big(f(f(n))\big)^2 \leq f(n) \cdot \big(f(f(n))\big)^2 \leq n(n^2+2n) < n(n+1)^2$$

so $f(f(n)) < n+1$, but $f(n) \geq n$ implies $f(f(n)) \geq n$, so $f(f(n)) = n$. Now

$$f(n)^2(f(n)-1) \leq f(f(n)) \cdot \big(f(f(f(n)))\big)^2 = nf(n)^2$$

hence $n \leq f(n) \leq n+1$ (two possible values for $f(n)$).
If $f(n) = n+1$ then $f(n+1) = f(f(n)) = n$, and so $C_{n+1} = C_n = C_{n-1} \cup \{n\}$ are good (because $n-1 \notin C_{n-1}$), hence both $P(n), P(n+1)$ are satisfied with $n+1 \in B$. And if $f(n) = n$ then $C_n = C_{n-1}$ is good, and so $P(n)$ is satisfied with $n \in B$.
Now $P(0), P(1), P(2), ...$ are all satisfied, so C is good and

$$f(n) = \begin{cases} n+1 & \text{if } n \in C \\ n-1 & \text{if } n-1 \in C \\ n & \text{otherwise} \end{cases}$$

as desired (1st Type). $Case2:\ \ f(0) = f(1)$. Then like case1, we see that $f(f(0)) = 0$, so again, $f(0) \in \{0, 1\}$, but $f(0) = 1$ leads to $f(1) = 0 \neq f(0)$, a contradiction! so $f(0) = f(1) = 0$, and hence by $(*)$ we see that $f(n) > 0\ \forall\, n \geq 2$. If $f(n) = 1$ for some $n \geq 2$ then

$$0 < n^3 - n^2 \leq f(n) \cdot \big(f(f(n))\big)^2 = f(1)^2 = 0$$

a contradiction! So $f(n) \geq 2$ whenever $n \geq 2$. Now we define

$$B_2 := \{n \geq 2 \mid (f(2), f(3), ..., f(n)) \text{ is a permutation of } (2, 3, ..., n)\}$$

and $C_2 = \{n \geq 2 \mid n \notin B_2\}$. Then just similar to case1, using $(*)$ instead of injectivity, we prove that C_2 is good and

$$f(n) = \begin{cases} 0 & \text{if } n \in \{0,1\} \\ n+1 & \text{if } n \in C_2 \\ n-1 & \text{if } n-1 \in C_2 \\ n & \text{otherwise} \end{cases}$$

as desired (2nd Type). So we are done. □

Problem A7. Let $n > 1$ be an integer. Determine the smallest positive real m such that, for every n reals $x_1, x_2, \ldots, x_n \in [0,1]$, there is a permutation $(y_1, y_2, \ldots, y_n)$ of $(x_1, x_2, \ldots, x_n)$, such that, letting $y_{n+1} = y_1$,

$$\sum_{k=1}^{n} \sqrt{|y_{k+1} - y_k|} \leq m$$

Proposed by *Luigi Amedeo Bianchi & Leonardo Franchi, Italy*

Solution. Answer. $m = \sqrt{n-1} + 1$.

Attainability.

Take $y_1 \leq y_2 \leq \ldots \leq y_n$, so that $\sum_{k=1}^{n-1} |y_{k+1} - y_k| \leq 1$.

$$\sum_{k=1}^{n} \sqrt{|y_{k+1} - y_k|} \leq \sum_{k=1}^{n-1} \sqrt{|y_{k+1} - y_k|} + 1 \leq \sqrt{n-1} + 1$$

by an application of AM-QM on $|y_{k+1} - y_k|$, for $k = 1, 2, \ldots, n-1$.

Optimality.

Take $x_k = \frac{k-1}{n-1}$. We will show that, for every permutation $(y_1, y_2, \ldots, y_n)$, the sum in question is at least $\sqrt{n-1} + 1$.

Suppose, without loss of generality, that $y_1 = 0$. Let m be the index such that $y_m = 1$. Let $z_k = |y_{k+1} - y_k|$. By triangle inequality, $z_1 + z_2 + \ldots + z_{m-1} \geq 1$ and $z_m + \ldots + z_n \geq 1$.

We will prove that $\sum_{k=1}^{n} \sqrt{z_k} \geq \sqrt{n-1} + 1$ in two ways.

For our fist approach, we show a more general fact: given $a_1, a_2, \ldots, a_n$ reals with sum at least 2 and $a_k \geq \frac{1}{n-1}$, then $\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n} \geq \sqrt{n-1} + 1$.

Suppose that the a_k's are in increasing order. We will apply the following procedure to them. Let j be the smallest index for which $a_j > \frac{1}{n-1}$. If $j = n$ we don't do anything. Otherwise, we replace a_j with $\frac{1}{n-1}$ and a_n with $a_n + a_j - \frac{1}{n-1}$. By concavity of the square root, $1 + \sqrt{a_n + a_j - 1} < \sqrt{a_n} + \sqrt{a_j}$, thus $\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n}$ decreases, while all other hypotheses still hold.

Since after each step one more term becomes equal to $\frac{1}{n-1}$, the process will eventually stop with $a_1 = a_2 = \ldots = a_{n-1} = \frac{1}{n-1}$ and $a_n \geq 1$, for which the sum of square roots is at least $\sqrt{n-1} + 1$.

Alternatively, we can observe that

$$\left(1 - \frac{1}{n-1}\right) \sqrt{z_k} \geq \frac{1 - z_k}{\sqrt{n-1}} + \left(z_k - \frac{1}{n-1}\right)$$

by concavity of the square root function (since $\frac{1}{n-1} \leq z_k \leq 1$). Now summing for all values of k we get $\sum_{k=1}^{n} \sqrt{a_k} \geq \sqrt{n-1} + 1$. □

Another solution. We claim that the minimum is $m = 1 + \sqrt{n-1}$.

First note that when $x_i = \frac{i-1}{n-1}$ for every $1 \leq i \leq n$, one cannot find a permutation that gives a smaller sum. For this, it is sufficient to note that in the permutation, if $y_1 = 0$ and $y_k = 1$, the smallest value is obtained if $y_1, y_2, \ldots, y_k$ is increasing and $y_k, y_{k-1} \ldots, y_n$ is decreasing. In that case, every difference (in absolute value) is of the form $\frac{a_i}{n-1}$, with all a_i being at least one and summing to $2(n-1)$. Since $x \to \sqrt{x}$ is a concave function, by the inequality of Karamata, it is at least equal to the case where all a_i are 1, except from one value being $n-1$.

On the other hand, in general orden them as $y_1 \leq y_2 \leq \ldots \leq y_n$.

Then by Jensen's inequality, we have that $\sum_{i=1}^{n-1} \sqrt{y_{i+1} - y_i} \leq (n-1)\sqrt{\frac{y_n - y_1}{n-1}} \leq \sqrt{n-1}$, which together with $\sqrt{y_n - y_1} \leq 1$ implies the result.

Problem A8. Initially, there is a non constant polynomial $P(x)$ with complex coefficients with nonzero constant term on the board. At each step, we can choose a complex number C and write $P(Cx)$ are $CP(x)$ on the board. If two (not necessarily distinct) polynomials $P(x), Q(x)$ are on the board, then we can write their composition or product on the board.

i. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of complex numbers, we call a polynomial $f(x)$ with complex coefficients nice if $f(A) = B$. Find all polynomials $P(x)$ that can be written on the board initially that for every two sets A, B, and such that after some of the above operations we can produce a nice polynomial.

ii. Let $P(x)$ has real coefficients and sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of real numbers. Find all polynomials $P(x)$ that can be written on the board initially that for every two sets A, B, and such that after some of the above operations we can produce a nice polynomial.

Proposed by *Navid Safaei, Iran*

Solution. Let $P(x) = Q(x^k)$ far some positive integer k. Then, all the above-mentioned operations guarantees that the resulting polynomial is indeed a polynomial in x^k. Assume that a_1, a_2 be two distinct roots of $x^k - c$ for some c. That is, $a_1^k = a_2^k = c$. Then, the resulting polynomial will have the same values at a_1, a_2, impossible.

Now we prove that if $P(x) \neq Q(x^k)$ the polynomial $P(x)$ is indeed nice. At first we prove that if there is a polynomial $R(x)$ such that $R(a_1) = b_1, R(a_2) = b_2$ then we can produce a nice polynamial. Let $a, b \in A$ and $T(x)$ be a polynamial that $T(a) = 0, T(b) = 1$ then for all $b \in A$ define $P_b(x) = \prod_{a \in A - \{b\}} T(x)$. Then $P_b(x) = \{ \begin{matrix} 1 & x = b. \\ 0 & x \in A - \{b\}. \end{matrix}$ Hence, defining .

$$R(x) = \prod_{b \in A} S(P_b(x))$$

Where $S(x)$ is a polynomial with $S(0) = 1, S(1) = R(b)$.

Back to our problem, it suffices to prove that there is a polynomial $R(x)$ such that $R(a_1) = b_1, R(a_2) = b_2$ with $a_1 \neq a_2$. Without loss of generality, assume that $a_1 \neq 0$, then we shall prove that the polynomial $C.P(B.P(Ax))$ works for some complex numbers A, B, C. If r is a root of $P(x)$ such that $\frac{a_2 r}{a_1}$ is also a root then $r, \frac{a_2 r}{a_1}, \ldots, (\frac{a_2}{a_1})^k r, \ldots$ must all be roots. Hence, $(\frac{a_2}{a_1})^k = 1$, for some $k \geq 2$. It follows that $P(x) = Q(x^k)$.

Hence, there is at least one root r of $P(x)$ such that $\frac{a_2r}{a_1}$ is not the root of $P(x)$. Now, we define the following procedure, as well:

$$a_1 \underset{\omega_r^r}{\rightarrow} r \underset{a_1}{\rightarrow} P(0)\times \underset{\times B}{\rightarrow} 0 \underset{P}{\rightarrow} P(0) \underset{\times \frac{w_1}{P(0)}}{\underset{\sim}{w}} b_1.$$

And then;

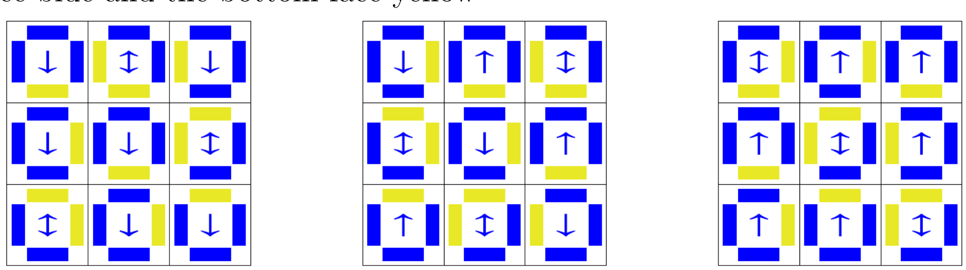
Now consider the root s of the equation $P(x) = \frac{b_2}{b_1}P(0)$ and then define $B = \frac{s}{P(\frac{a_2r}{a_1})}$ then $P(BP(\frac{a_2r}{a_1})) = \frac{b_2}{b_1}P(0)$ and hence $\frac{b_1}{P(0)} \cdot \frac{b_2}{b_1}P(0) = b_2$. Finally, if $b_1 = 0$ then setting $B = \frac{b_2}{P(\frac{a_2r}{a_1})}$ and finish the two steps. Solution ii. It is easy to find that $P(x) \neq Q(x^2)$. We also prove that $P(x)$ must at least one single root otherwise it doesn't change its sign at all and hence no polynomial after the above-mentioned operation can't take two numbers of the different signs. Now, we shall prove if $P(x) \neq Q(x^2)$ and $P(x)$ has at least one single root then $P(x)$ is indeed nice. It is easy to find that $P(x)$ has a root r such that $\frac{a_2r}{a_1}$ is not the root of $P(x)$. Now, at least one of the equations $P(x) = \frac{b_2}{b_1}P(0)$ or $P(x) = -\frac{b_2}{b_1}P(0)$ has a real root. Then, if the equation $P(x) = -\frac{b_2}{b_1}P(0)$ has a real root then we shall send (a_1, a_2) to $(b_1, -b_2)$. Now, if in the third transformation, we change B by $-B$, we are done.$\square$

8. Solutions-Combinatorics

Problem C1. Let a *brick* be a unit cube such that its three faces sharing a common vertex are colored blue, its two faces are colored yellow, and the remaining face is colored green. Find the largest positive integer n such that n^3 bricks (not necessarily identical ones) can be stacked to form a cube of side n, where for every two neighboring bricks their joining faces are either colored green in the both bricks, or colored blue in one brick and yellow in the other brick.

Proposed by *Bojan Basic, Armenia*

Solution. Note that, in the arranged cube, there is a total of $3n^3$ blue squares (3 on each brick), while there can be at most $6n^2$ blue squares on the surface of the arranged cube, which means that there are at least $n^2(3n-6)$ blue squares in the interior of the arranged cube. However, since each interior blue square is paired with one yellow square, and there is a total of $2n^3$ yellow squares, we obtain the inequality $n^2(3n-6) \leqslant 2n^3$, from which follows $n \leqslant 6$. Let us now prove that $n = 6$ can be reached. We first note that 8 bricks can be stacked to form a cube of side 2 with four completely blue faces and two completely yellow faces, where the two yellow faces are neighbors. Indeed, we can first take two bricks and place them next to each other with their joining faces being both green, and the remaining faces colored in a mirrored way (where the green faces determine the mirroring plane). Then the required cube of side 2 can be obtained from four such $2 \times 1 \times 1$ parallelepipeds (using only translations). Therefore, the problem is reduced to showing that 27 cubes from the previous paragraph can be stacked to form a cube of side 6 (where for every two neighboring cubes their joining faces are colored blue in one cube and yellow in the other one). One possible formation is shown in the picture below. The left-hand side represents the bottom 3×3 layer, the middle part represents the middle layer, and the right-hand side represents the top layer. Cubes marked by $\updownarrow$ have their both the top and the bottom face colored blue, cubes marked by $\downarrow$ have their bottom face blue and the top face yellow, while cubes marked by $\uparrow$ have their top face blue and the bottom face yellow



□

Problem C2. A mathematician born in the 20[th] century noted:

> My phone number has six digits. The three-digit number formed by the first three digits is different than the three-digit number formed by the last three digits. The first three digits are nonzero and mutually different, and if we consider all the other five three-digit integers that

> can be obtained by their permutation, the sum of them five gives my birth year. The last three digits are also nonzero and mutually different, and if we consider all the other five three-digit integers that can be obtained by their permutation, the sum of them five gives the birth year of my twin brother.

Was this mathematician born during a weekend?

Proposed by *Bojan Basic, Armenia*

Solution. Let $\overline{abc}$ be the three-digit number formed by the first three digits, and $\overline{a'b'c'}$ be the three-digit number formed by the last three digits. In all the 6 permutations of the digits of the number $\overline{abc}$, each of the digits a, b and c will be in the first position twice, in the second position twice, and in the third position twice; therefore, the sum of all six numbers obtained that way will equal $2 \cdot 100(a+b+c) + 2 \cdot 10(a+b+c) + 2(a+b+c)$, that is, $222(a+b+c)$. Exempting the number $\overline{abc}$, the sum of the remaining five numbers will equal $222(a+b+c) - \overline{abc}$. By the conditions of the problem, this gives the birth year of the mathematician from the statement. Similarly, $222(a'+b'+c') - \overline{a'b'c'}$ gives the birth year of his twin brother.

We can assume, w.l.o.g., $\overline{abc} > \overline{a'b'c'}$ (otherwise we swap the roles of the mathematician and his brother). Clearly, the mathematician and his brother were born either in the same year, or in two consecutive years (if one of them is born on December 31st a few minutes before midnight, and the other one a few minutes after midnight, on January 1st). That means $222(a'+b'+c') - \overline{a'b'c'} = 222(a+b+c) - \overline{abc} + \delta$, with $\delta \in \{-1, 0, 1\}$. We can rewrite this as $\overline{abc} = \overline{a'b'c'} + 222k + \delta$, that is,

$$100a + 10b + c = 100(a' + 2k) + 10(b' + 2k) + c' + 2k + \delta,$$

for $k = a + b + c - a' - b' - c'$ (and we have $k \in \{1, 2, 3\}$, because of $\overline{abc} > 222k + \delta + \overline{a'b'c'} \geqslant 222k + 123$). It now follows $c \equiv c' + 2k + \delta \pmod{10}$, then $b \equiv b' + 2k + \varepsilon_1 \pmod{10}$ for $\varepsilon_1 = \lfloor \frac{c'+2k+\delta}{10} \rfloor$ (and clearly $\varepsilon_1 \in \{0, 1\}$), as well as $a \equiv a' + 2k + \varepsilon_2 \pmod{10}$ for $\varepsilon_2 = \lfloor \frac{b'+2k+\varepsilon_1}{10} \rfloor$ (and clearly $\varepsilon_2 \in \{0, 1\}$). It is impossible that $\varepsilon_1 = \varepsilon_2 = 0$, since then the congruences from the previous sentence would be equalities, and then would follow $k = a + b + c - a' - b' - c' = a' + 2k + b' + 2k + c' + 2k + \delta - a' - b' - c' = 6k + \delta$, which is a clear contradiction. Adding these three congruences together, we get

$$a + b + c \equiv a' + b' + c' + 6k + \varepsilon_1 + \varepsilon_2 + \delta \pmod{10}.$$

Now, because of $a + b + c - a' - b' - c' = k$, the relation above reduces to $k \equiv 6k + \varepsilon_1 + \varepsilon_2 + \delta \pmod{10}$, that is, $5k + \varepsilon_1 + \varepsilon_2 + \delta \equiv 0 \pmod{10}$. From this follows that $5 \mid \varepsilon_1 + \varepsilon_2 + \delta$, and since at least one among $\varepsilon_1, \varepsilon_2$ equals 1, we conclude $\delta = -1$ and $\{\varepsilon_1, \varepsilon_2\} = \{0, 1\}$; then, $5k \equiv 0 \pmod{10}$, and we thus conclude $k = 2$.

Therefore, we have

$$\overline{abc} = \overline{a'b'c'} + 443,$$

where, when performing the addition, there is a carry in exactly one column (ε_1 and ε_2 are exactly flags of the existence of carries in the corresponding columns). Since the mathematician was born in the 20th century, the birth years of him and his brother have to be in the range from 1899 to 2001. Therefore,

$$1899 \leqslant 222(a'+b'+c') - \overline{a'b'c'} \leqslant 2001.$$

Since $\overline{a'b'c'} = \overline{abc} - 443 \leqslant 987 - 443 = 544$, we have $222(a'+b'+c') \leqslant 2001 + 544 = 2545$, from where $a'+b'+c' \leqslant 11$. Further, if $\varepsilon_1 = 1$, then $c' \geqslant 7$ (since we need a carry in the last column), and actually $c' \geqslant 8$ because $c' = 7$ is impossible as that would imply $c = 0$; together with $a', b' \geqslant 1$ and the fact that a' and b' are different, in this case we get $a'+b'+c' \geqslant 8+1+2 = 11$. In the case $\varepsilon_2 = 1$ we have $b' \geqslant 7$ (via $b' \geqslant 6$ and $b' \neq 6$ as that would imply $b = 0$), from where $a'+b'+c' \geqslant 7+1+2 = 10$. Altogether, we anyway have $a'+b'+c' \in \{10, 11\}$.

Suppose first that $a'+b'+c' = 11$. Then $1899 \leqslant 222 \cdot 11 - \overline{a'b'c'} \leqslant 2001$, from where $441 \leqslant \overline{a'b'c'} \leqslant 543$. Therefore, having in mind $a' \geqslant 4$, and one of $c' \geqslant 8$ or $b' \geqslant 7$ (as we have seen in the previous paragraph), and since the remaining digit is at least 1, it follows that $a'+b'+c' \geqslant 7+4+1 = 12$, a contradiction.

Finally, $a'+b'+c' = 10$. Then $1899 \leqslant 222 \cdot 10 - \overline{a'b'c'} \leqslant 2001$, from where we get $219 \leqslant \overline{a'b'c'} \leqslant 321$. Since we have already seen that in the case $\varepsilon_1 = 1$ it is impossible to have $a'+b'+c' = 10$, there remains $\varepsilon_2 = 1$; we have also seen that then $b' \geqslant 7$, and since the obtained bounds for $\overline{a'b'c'}$ imply $a' \geqslant 2$, the only possibility is $b' = 7$, $a' = 2$ and $c' = 1$, that is, $\overline{a'b'c'} = 271$. Then we evaluate $222(a'+b'+c') - \overline{a'b'c'} = 222 \cdot 10 - 271 = 1949$, then $\overline{abc} = 271 + 443 = 714$, and finally $222(a+b+c) - \overline{abc} = 222 \cdot 12 - 714 = 1950$.

Therefore, the mathematician and his brother were born in the night between the years 1949 and 1950. Since July 1st, 2023, is Saturday, and since $365 \equiv 1 \pmod 7$, and since 73 years passed between 1950 and 2023, among which 18 were leap years, noticing that $73 + 18 = 91 \equiv 0 \pmod 7$, we conclude that July 1st, 1950, was Saturday. Therefore, January 1st, 1950, was Sunday (as $3 \cdot 31 + 28 + 2 \cdot 30 = 181$ days passed by till July 1st, and $181 \equiv -1 \pmod 7$). Therefore, the mathematician and his brother were born during the night between Saturday and Sunday, that is, they were born during a weekend. □

Problem C3. Is it possible for a convex polygon to be partitioned into finitely many smaller convex polygons such that each of these smaller polygons has at least 6 sides and each vertex of these smaller polygons is also a vertex of at least two other smaller polygons?

Proposed by *Liam Baker, South Africa*

Solution. We prove that it is not possible to do so.

Let q denote the number of vertices in the larger polygon, let j denote the number of interior vertices created by the partitioning, and let g_n denote the number of smaller polygons with n sides. Then by adding up the angles around each interior point and the angles interior to the larger polygon, we get $2\pi j + \pi(q-2)$. This sum of angles can also be calculated by adding up the angles in each smaller polygon, giving a sum of $\sum_n \pi(n-2)g_n$. Thus we get that $\sum_n (n-2)g_n = 2j + q - 2$.

Let us also count the number of interior and boundary vertices; by adding up the number of vertices in each smaller polygon, we get $\sum_n ng_n$. However, since each vertex is the vertex of at least 3 smaller polygons, each vertex is being counted at least thrice here; hence we get that $\sum_n ng_n \geq 3j+3q$. Thus

$$3j+3q \leq \sum_n ng_n = \sum_n \frac{n}{n-2}(n-2)g_n \leq \frac{6}{4}\sum_n (n-2)g_n = \frac{3}{2}[2j+q-2] = 3j+\frac{3}{2}q-3,$$

a contradiction. □

Problem C4. A *binoku* is a 9×9 grid of zeros and ones such that each row and each column and each of the nine 3×3 subgrids contain at least one 0 and at least one 1. An *incomplete binoku* is a 9×9 grid containing zeros, ones, and empty cells. What is the largest number of empty cells that an incomplete binoku can contain if it can be completed into a binoku in a unique way?

Proposed by *Stijn Cambie, South Korea*

Solution. We claim that there at most 18 empty cells in an incomplete binoku that can be completed in a unique way. The binoku on the right contains 18 zeros, each of which is the unique zero in either its row, its column, or its box (i.e. its 3×3 subgrid), so the incomplete binoku obtained by emptying each of these 18 cells clearly has a unique solution (i.e. a unique binoku into which it can be completed). This shows that the upper bound can be attained.

To establish the upper bound, we begin by noticing that, for each empty cell in an incomplete binoku with a unique solution, the incomplete binoku with a single empty cell that is obtained from the solution by emptying that cell also has a unique solution. If the other cells in the row of this empty cell contain both zeros and ones, and the other cells in its column and the other cells in its box do so, too, then adding a 0 or a 1 to the empty cell both yield a binoku, contradicting uniqueness of the solution. Hence the other cells of either the row or the column or the box of the empty cell all contain the same number.

0	0	1	1	1	0	1	1	0
1	1	0	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1
1	1	0	0	0	1	1	1	0
1	1	1	1	1	0	1	1	1
1	1	1	1	1	0	1	1	1
1	1	0	1	1	0	0	0	1
1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	0

□

We call a cell of a binoku *delible* if it contains the only 0 or the only 1 in its row or column or box. Each empty cell in an incomplete binoku of which this binoku is the unique solution is therefore one of these delible cells. By definition, we can assign to each delible cell a row or a column or a box contaning it and in which all the other entries are different from the entry of that cell. This assignment is not necessarily unique (since a cell may, for example, contain the only 0 both in its row and in its column), but the following holds:

Claim 1. No row, column, or box can be associated to more than one delible cell.

Proof. Suppose to the contrary that two delible cells are associated to the same row, column, or box. Without loss of generality, the first cell is 0, so all the other cells (and in particular the second delible cell) in this row, column, or box are 1. Because the second delible cell is 1, all the other cells in this row, column, or box are 0, so any third cell is both 0 and 1, which is a contradiction. □

This observation extends to yield the following:

Claim 2. If a row or column associated with a delible cell contains another delible cell, then the latter cannot be associated with a box.

Proof. Suppose to the contrary that it is associated with a box. The intersection of the box and row or column contains three cells, the delible cell associated to the box (which is 0 without loss of generality) and two other cells (which are therefore 1). Since they do not therefore all have the same value, the delible cell associated with the row or column must be among these two cells, so these two cells must have different values, which is a final contradiction. □

A similar argument proves

> *Claim 3.* A box associated to a delible cell does not contain any other delible cell.

Finally, we observe the following:

> *Claim 4.* Any box contains at most 3 delible cells associated to a row and at most 3 delible cells associated to a column.
>
> *Proof.* If a box contained 4 delible cells associated to a row (or column), there would be two delible cells associated to the same row (or column), contradicting *Claim 1.* □

Now let b, c, r denote the number of boxes, columns, and rows of a binoku that are associated to a delible cell in this way. By *Claim 1*, $b+c+r$ is the number of delible cells and hence the largest possible number of empty cells in an incomplete binoku of which this binoku is the unique solution. Now each of the r rows associated to a delible cell has 9 cells and each of the c columns associated to a delible cell contains $9-r$ cells not part of one these r rows. Finally, the delible cell in each of the b boxes is not in any of these r rows or c columns by *Claim 2*. There are 81 cells in total, so

(1)
$$9r + (9-r)c + b \leq 81 \quad \Longrightarrow \quad 81 - (b+c+r) \geq 8(c+r) - cr \geq 8(c+r) - \frac{(c+r)^2}{4}$$

by the inequality between arithmetic and geometric means. Suppose to the contrary that $b+c+r \geq 19$. Then Equation (1) yields $f(c+r) \leq 248$, where $f(z) = 32z - z^2$, which is increasing for $z \leq 16$ and decreasing for $z \geq 16$. Since $f(14) = f(18) = 252$, this implies that $c+r \leq 13$ or $c+r > 18$. Trivially, $c, r \leq 9$, so the latter is a contradiction. Thus $c+r \leq 13$ and hence $b \geq 6$ for $b+c+r \geq 13$.

Now suppose that any of the $9-b$ boxes not associated to a delible cell contains at most 4 delible cells. Each of these must be assigned to a row or column, so

(2)
$$c + r \leq 4(9-b) \quad \Longrightarrow \quad (b+c+r) + 3b \leq 36.$$

If $b+c+r \geq 19$, this implies $3b \leq 17$, so $b \leq 5$, which is a contradiction. Hence there is a box B, not associated to a delible cell by *Claim 3*, that contains at least 5 delible cells. Hence, and without loss of generality, at least 3 of them are associated with rows. These rows are different by *Claim 1* and they are shared by box B with two other boxes, B_1, B_2. By *Claim 2*, B_1, B_2 are not associated with any delible cell either. In particular, this shows that $b \leq 9-3 = 6$. By the above, it follows that $b = 6$ and hence $c+r = 13$ for $b+c+r \geq 19$. Moreover, by *Claim 3*, the $b = 6$ boxes associated to a delible cell do not contain any other delible cell. Finally, box B contains at most $3+3 = 6$ delible cells by *Claim 4*, while boxes B_1, B_2 contain at most 3 cells each associated to one of their columns by *Claim 4* again, but do not contain any delible cells associated to one of their rows by *Claim 1* because that row is already associated with a delible cell in box B. This proves that $c+r \leq 6+3+3 = 12$. This is the final contradiction showing that $b+c+r \leq 18$. This establishes the required bound and completes the solution. □

Problem C5. In a plane, 2022 points are colored either black or white, in such a way that no three points lie on the same line, and that the triangle formed by every three black points contains at least one white point. What is the largest possible number of black points?

Proposed by *Art Waeterschoot, Belgium*

Solution. The largest number of black points is 1012.

Proof of the upper bound. Let N be the number of black points and let v be a black point. Then v is a vertex of at least $N-2$ triangles formed by black points whose interiors are disjoint. This means that there are at least $N-2$ white points, so that $2N-2 \leq 2022$ and $N \leq 1012$.

Proof of the lower bound. Take the black points to be the vertices of a convex 1012-gon. The lines that join the vertices divide the polygon into a number of regions. Choose a side s of the polygon. For every vertex v not on s, place a white point in the region of the polygon that meets v and that lies in the triangle formed by s and v. This gives a total of 2022 points. We claim that this construction is valid. Let v_1 and v_2 be the vertices of s, in clockwise order, and let t_1, t_2, t_3 be vertices of the polygon. We show that $\triangle t_1t_2t_3$ contains a white point. If s is a side of the triangle, we are done, by construction. Assume now, by changing the numbering if necessary, that t_1 and t_2 are not vertices of s, that v_1, v_2, t_1, t_2 lie in clockwise order and that t_1, t_2, t_3 lie in clockwise order. Then t_3 lies either strictly between t_2 and v_2 or strictly between v_1 and t_1. In the first case, $\triangle t_1t_2t_3$ contains the white point in $\triangle t_2v_1v_2$. In the second case, $\triangle t_1t_2t_3$ contains the white point in $\triangle t_1v_1v_2$. □

Problem C6. In a network of 60 metro stations $1, 2, \ldots, 60$, there are direct connections $C_{i,j}$ between some stations $i < j$. On such a connection, one can travel in either direction for one pound. Let a_i be the number of stations one can travel to for one pound starting from station i (station i not included). For any connection $C_{i,j}$, let $L_{i,j}$ be the number of stations that are (strictly) cheaper to travel to from station i than from station j. Note that station i is one such station. For any connection $C_{i,j}$, let $H_{i,j}$ be the number of stations that are more expensive to travel to from station i than from station j. Note that station j is one such station. What is the maximum possible value of

$$\sum_{C_{i,j} \in E} (a_i + a_j) \cdot L_{i,j} \cdot H_{i,j}$$

where E is the set of all connections?

Proposed by *Stijn Cambie, South Korea*

Solution. First, we make the following crucial observation.

Lemma 8.1. *For any connection $C_{i,j} \in E$, we have $L_{i,j} \leq 60 - a_j$ and $H_{i,j} \leq 60 - a_i$.*

Proof of Lemma 8.1. Note that every station k different from j for which there is connection between i and k, it is not more expensive to travel to from station i than from station j. There are $a_i - 1$ of them. Also station i is not in the set of stations counted by $H_{i,j}$ and hence $H_{i,j} \leq 60 - a_i$. Analogously $L_{i,j} \leq 60 - a_j$.■

By double counting, we find the following two equalities (the first one being the hand shaking lemma)

Lemma 8.2. *We have*

$$\sum_i a_i = 2|E| \ \textit{and} \ \sum_i a_i^2 = \sum_{C_{i,j}\in E} a_i + a_j$$

As a last observation, note that by AM-GM we have $x(60-x) \leq 30^2$ for every x and hence

$$(a_i+a_j)(60-a_j)(60-a_i) = a_i(60-a_i)\cdot(60-a_j)+a_j(60-a_j)\cdot(60-a_i) \leq 30^2(120-a_i-a_j).$$

Applying these three observations, we find

$$\begin{aligned}\sum_{C_{i,j}\in E}(a_i+a_j)\cdot L_{i,j}\cdot H_{i,j} &\leq \sum_{C_{i,j}\in E}(a_i+a_j)(60-a_j)(60-a_i)\\ &\leq 30^2\sum_{C_{i,j}\in E}(120-a_i-a_j)\\ &\leq 30^2\sum_{i=1}^{60}(60-a_i)a_i \text{ by (lemma 2)}\\ &\leq 2\cdot 30^5.\end{aligned}$$

Equality is possible if for all connections $e = C_{i,j}$ we have $L_{i,j} = H_{i,j} = a_i = a_j = 30$. This is satisfied if there are connections $C_{i,j}$ exactly when $1 \leq i \leq 30$ and $31 \leq j \leq 60$. □

Problem C7. There are $n!$ baskets in a row, numbered $1, 2\dots, n!$. John first puts a stone in every basket. John then puts 2 stones in every second basket. John repeats this until putting n stones into every nth basket. In other words, for each $i = 1, 2, \dots, n$, John puts i stones into the baskets labeled $i, 2i, 3i, \dots, n!$.

Let x_i be the number of stones in basket i after all these steps. Show that

$$n!\cdot n^2 \leq \sum_{i=1}^{n!} x_i^2 \leq n!\cdot n^2\cdot\sum_{i=1}^{n}\frac{1}{i}.$$

Proposed by *Kaarel Hänni, Estonia*

Solution. Note that on the ith step, John places a total of $i\frac{n!}{i} = n!$ stones into the baskets. Hence, after all n steps, there are $n!n$ stones in the baskets. The QM-AM inequality then gives

$$\sum x_i^2 \geq \frac{\left(\sum x_i\right)^2}{n!} = n!n^2,$$

proving the first desired inequality. For the other inequality, note that $x_i = \sum_{j|i} j$, and hence

$$\sum_i x_i^2 = \sum_i \sum_{(j,k)\in[n]^2,\ j|i,\ k|i} jk.$$

For fixed j and k, we have jk appearing in the index i term of the first sum if and only if $j|i$ and $k|i$, which in turn happens if and only if $\operatorname{lcm}(j,k)|i$. The number of such $i \in \{1, \dots, n!\}$ is exactly $\frac{n!}{\operatorname{lcm}(j,k)}$. Hence, switching the order of summation to sum first by (j,k), we get

$$\sum_i x_i^2 = \sum_{(j,k)\in[n]^2} jk\frac{n!}{\operatorname{lcm}(j,k)} = n! \sum_{(j,k)\in[n]^2} \gcd(j,k).$$

Now note that for some fixed $i \in [n]$ to be $\gcd(j,k)$, it is necessary (but not sufficient) that $i|j$ and $i|k$. Hence, for a fixed $i \in [n]$, the number of such pairs $(j,k) \in [n]^2$ is at most $\frac{n^2}{i^2}$, and so

$$\sum_i x_i^2 \leq n! \sum_{i=1}^{n} i\frac{n^2}{i^2} = n!n^2 \sum_{i=1}^{n} \frac{1}{i}.$$

□

Problem C8. A rectangular grid is divided by two perpendicular straight lines into four smaller rectangles with integral side lengths. It is possible to remove one among these four rectangles in such a way that the remaining figure can be exactly covered by rectangles of size 2×3 and 3×2. Prove that it is possible to exactly cover one among these four smaller rectangles by rectangles of size 2×3 and 3×2. (By *exact covering* we mean covering without gaps, overlaps and overflows.)

Proposed by *Oleg Kosik, Estonia*

Solution. Call a figure *coverable* if it can be exactly covered by rectangles of size 2×3 and 3×2. Let the original rectangle be of size $(a+d) \times (b+c)$ and let the figure remaining after cutting out the upper left corner of size $a \times b$ be coverable. This would be impossible if $c = 1$ or $d = 1$, whence we can assume in the rest that $c \geq 2$ and $d \geq 2$. Coverability implies that $6 \mid ac + bd + cd$. Also one can notice that a rectangle of integral side lengths $n \times m$ where $n \geq 2$ and $m \geq 2$ is coverable if and only if $6 \mid nm$. Indeed, rectangles of size $2k \times 3l$ and $3l \times 2k$ are obviously coverable, whereas rectangles of size $6k \times (6l \pm 1)$ can be divided into rectangles of size $6k \times 3$ ja $6k \times 2s$, both of which are coverable by the above; the symmetric case is analogous.

Since $2 \mid ac+bd+cd$, at least one of a, b, c and d must be even. We show next that at least one of a, b, c and d must be divisible by 3. For that, suppose the contrary, i.e., none of a, b, c and d being divisible by 3. If either $3 \mid a + d$ or $3 \mid b + c$ then $3 \mid bd$ or $3 \mid ac$, respectively, because $3 \mid ac + bd + cd$. Since 3 is prime, we obtain a contradiction. Hence $3 \nmid a + d$ and $3 \nmid b + c$. Thus either $a \equiv d \equiv 1 \pmod 3$ or $a \equiv d \equiv 2 \pmod 3$, as well as either $b \equiv c \equiv 1 \pmod 3$ or $b \equiv c \equiv 2 \pmod 3$. We get contradiction in all these cases:

- If $a \equiv b \equiv c \equiv d \equiv 1 \pmod 3$ then colour the squares of the figure by the rising diagonals with 3 colours. The numbers of squares of each colour are different, whereas each rectangle of size 2×3 or 3×2 covers exactly 2 squares of each colour.
- The case $a \equiv b \equiv c \equiv d \equiv 2 \pmod 3$ is similar.

- If $a \equiv d \equiv 1 \pmod 3$ and $b \equiv c \equiv 2 \pmod 3$ then colour the squares of the figure by the falling diagonals with 3 colours. The numbers of squares of each colour are different, whereas each rectangle of size 2×3 or 3×2 covers exactly 2 squares of each colour.
- The case $a \equiv d \equiv 2 \pmod 3$ and $b \equiv c \equiv 1 \pmod 3$ is symmetric.

It remains to show that one of the small rectangles is coverable whenever one of a, b, c and d is divisible by 2 and also one of a, b, c and d is divisible by 3. The following table shows for each case which of the small rectangles is coverable:

	$3 \mid a$	$3 \mid b$	$3 \mid c$	$3 \mid d$
$2 \mid a$	$a \times c$	$a \times b$	$a \times c$	$a \times c$
$2 \mid b$	$a \times b$	$d \times b$	$d \times b$	$d \times b$
$2 \mid c$	$a \times c$	$d \times b$	$d \times c$	$d \times c$
$2 \mid d$	$a \times c$	$d \times b$	$d \times c$	$d \times c$

One can show this by the following case study:

- If one of a and d is divisible by one of 2 and 3 and one of b and c is divisible by the other of 2 and 3 then the corresponding small rectangle is coverable. (Fills all squares outside the two long diagonals of the table.)
- If a is divisible by one of 2 and 3 and d is divisible by the other of 2 and 3 then, by $6 \mid ac + bd + cd$, we must have $6 \mid ac$. As $a \geq 2$ (by divisibility) and $c \geq 2$, the rectangle $a \times c$ is coverable. Similarly, if b is divisible by one of 2 and 3 and c is divisible by the other of 2 and 3 then $d \times b$ is coverable. (Fills all squares of the secondary diagonal of the table.)
- If $6 \mid a$ or $6 \mid b$ then $a \times c$ or $d \times b$, respectively, is coverable since $c \geq 2$ and $d \geq 2$. (Fills the first two squares of the main diagonal of the table.)
- If $6 \mid c$ or $6 \mid d$ then $d \times c$ is coverable since $c \geq 2$ and $d \geq 2$. (Fills the last two squares of the main diagonal of the table.)

9. Solutions-Geometry

Problem G1. Let $\mathbb{R}$ denote the set of the real numbers and let $\mathbb{R}^2$ denote the Euclidean plane. Find all functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that for every non-degenerate triangle ABC with orthocenter H, the following equation holds:

$$f(H) = \frac{1}{3}(f(A) + f(B) + f(C)).$$

Proposed by *Liam Baker, South Africa*

Solution. It is evident that any constant function will satisfy this equation. We now show that if f satisfies this condition, then it must be a constant function. Let A and B be any two distinct points; we will show that $f(A) = f(B)$.

(1) Let C be any point not on line AB, and let H be the orthocentre of triangle ABC. Then A is the orthocentre of HBC and B is the orthocentre of HCA, so that

$$3f(A) = f(H) + f(B) + f(C) \quad \text{and} \quad 3f(B) = f(H) + f(C) + f(A).$$

Subtracting one equation from the other then gives us that $f(A) = f(B)$, which finishes the proof.

(2) Let O be any point equidistant from A and B, let Γ be the circle centred at O passing through A and B, and let C and D be any two other points on this circle. Then since O is the circumcentre of both ACD and BCD we have that

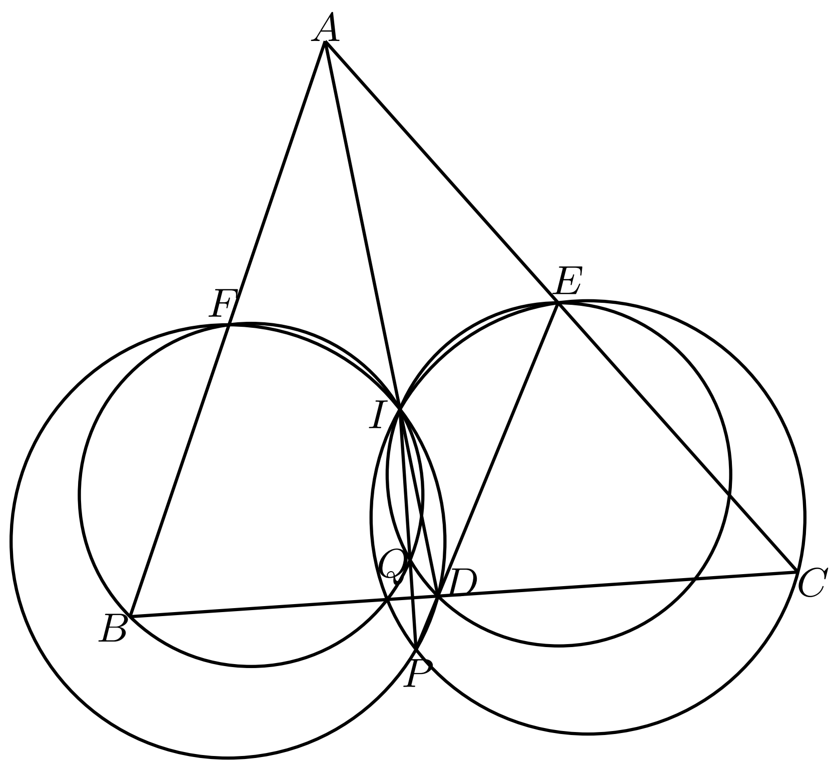
$$f(A) + f(C) + f(D) = 3f(O) = f(B) + f(C) + f(D) \implies f(A) = f(B),$$

which finishes the proof.

□

Problem G2. Let ABC be a triangle with incenter I and $AB \neq AC$. Let D be the intersection of line AI and line BC. Let E be the point on segment AC such that $CD = CE$. Similarly, let F be the point on segment AB such that the length of BF is equal to the length of BD. Let P be the intersection, that is different from I, of the circumcircle of the triangle CEI and the circumcircle of the triangle DFI. Similarly, let Q be the intersection, that is different from I, of the circumcircle of the triangle BFI and the circumcircle of the triangle DEI. Prove that PQ is orthogonal to BC.

Proposed by *Leonardo Franchi, Italy*



Solution.

We claim that P, Q and I are aligned on the perpendicular line from I to BC. In particular we are going to show that the line IP is perpendicular to BC and then thanks to the symmetry of the configuration the conclusion follows. Since I is incenter, we have by symmetry $ID = IE = IF$.

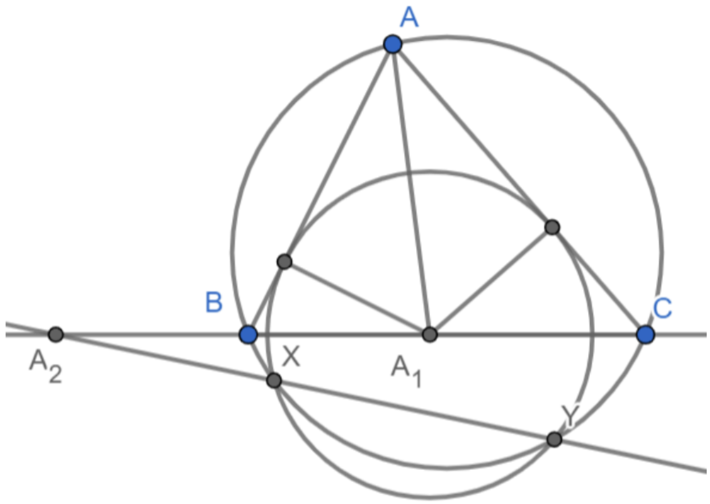
We now prove that P, D and E are collinear. In particular $\angle IPE = \angle ICA = \frac{\gamma}{2}$ since $PIEC$ is a cyclic quadrilateral. On the other hand $\angle IPD = \angle IFD$ since $FIDP$ is a cyclic quadrilateral. Now, since $\angle IFD = \angle IDF$, we get that $\angle IDF = \angle ADB - \angle FDB = \frac{\alpha}{2} + \gamma - (90 - \frac{\beta}{2}) = \frac{\gamma}{2}$ as we wanted.

Finally, since $\angle BDP = \angle EDC = 90 - \frac{\gamma}{2}$ and $IPD = \frac{\gamma}{2}$ the conclusion follows. □

Problem G3. Let ABC be a triangle with circumcircle Γ. Let ω_A be the circle such that the center of ω_A is on BC and ω_A is tangent to both AB and AC. Suppose that Γ and ω_A intersect at two distinct points, denoted by X_A and Y_A. Let A' be the intersection of the line BC and the line X_AY_A. Define B' and C' analogously. Prove that A', B' and C' are collinear.

Proposed by *Khakimboy Egamberganov, Uzbekistan*

Solution.



We will find the ratio $\frac{BA_2}{CA_2}$ and finish the problem using Menelaus Theorem and Ceva Theorem.

We will use a, b and c for sides BC, CA and AB, respectively. Note that AA_1 is bisector of $\angle BAC$ because the distances from A_1 to AB and AC are equal. By Bisector Theorem $BA_1 = \frac{ac}{b+c}$ and $CA_1 = \frac{ab}{b+c}$. Let r_A be the radius of ω_A. By area $S = \frac{br_A}{2} + \frac{cr_A}{2} \Leftrightarrow r_A = \frac{2S}{b+c}$.

By power of point, $A_2B \cdot A_2C = A_2A_1^2 - r_A^2 \Leftrightarrow (A_2A_1 - BA_1)(A_2A_1 + CA_1) = A_2A_1^2 - \frac{4S^2}{(b+c)^2} \Leftrightarrow A_2A_1^2 + A_2A_1(CA_1 - BA_1) - BA_1 \cdot CA_1 = A_2A_1^2 - \frac{4S^2}{(b+c)^2} \Leftrightarrow A_2A_1 \cdot \frac{a(b-c)}{b+C} - \frac{ab}{b+c} \cdot \frac{ac}{b+C} = -\frac{4S^2}{(b+c)^2} \Leftrightarrow A_2A_1 = \frac{a^2bc-4S^2}{a(b^2-c^2)}$

This way we calculate $A_2B = A_2A_1 - BA_1 = \frac{a^2bc-4S^2}{a(b^2-c^2)} - \frac{ac}{b+c} = \frac{a^2bc-4S^2-a^2c(b-c)}{a(b^2-c^2)} = \frac{a^2c^2-4S^2}{a(b^2-c^2)}$ and $A_2C = \frac{a^2b^2-4S^2}{a(b^2-c^2)}$

Taking the altitude AH_A from A, we have $S = \frac{a \cdot AH_A}{2} \Leftrightarrow 4S^2 = a^2 \cdot AH_A^2$ and using this on the previous equations: $A_2B = \frac{a^2\left(c^2-AH_A^2\right)}{a(b^2-c^2)} = \frac{a \cdot BH_A^2}{b^2-c^2}$ (using Pythagoras Theorem) and $A_2C = \frac{a \cdot CH_A^2}{b^2-c^2}$. The ratio $\frac{BA_2}{CA_2} = \frac{BH_A^2}{CH_A^2}$.

The points A_2, B_2 and C_2 are outside the sides (the points X and Y are on the arc BC without A and meets BC outside). By Menelaus, they are collinear if and only if $\frac{BA_2}{CA_2} \cdot \frac{CB_2}{AB_2} \cdot \frac{AC_2}{BC_2} = 1 \Leftrightarrow \left(\frac{BH_A}{CH_A} \cdot \frac{CH_B}{AH_B} \cdot \frac{AH_C}{BH_C}\right)^2 = 1$ and this last equation is true by Ceva Theorem on AH_A, BH_B and CH_C concurrent on the orthocenter H. $\square$

Problem G4. Let ABC be a triangle, and let its incircle touch the sides BC, CA, AB at points D, E, F, respectively. Let M and N be the midpoints of the segments DE and DF, respectively. Let P, that is different from B, be the intersection of the circumcircle of the triangle BDM and the circumcircle of the triangle ABC. Similarly, let Q, that is different from C, be the intersection of the circumcircle of the triangle CDN and the circumcircle of the triangle ABC. Prove that the points P, Q, M and N lie on a circle.

Proposed by *Khakimboy Egamberganov, Uzbekistan*

Solution. Let O, I and Ω be the circumcenter, the incenter and the circumcircle of ABC, respectively. Note that ω is the incircle of ABC. And, let ℓ be the radical axis of ω and Ω, i.e. locus of all points X satisfying $\text{Pow}_X(\omega) = \text{Pow}_X(\Omega)$.

First of all, we aim to show that the circumcircles of triangles BDM and CDN intersects the second time at point K ($K \neq D$) that lies again on the incircle ω. Actually, we do it more rigorously and show that this point K will be the tangent point of the incircle ω with a circle that passes through B and C.

Let $EF \cap BC = D'$ and X be the midpoint of the segment DD'. We know that $\{D', D; B, C\}$ is a harmonic bundle, therefore the circle with diameter DD' (let's say Γ, and see *Figure 1*) should be an Apollonian circle for points D and D' with coefficient $k = \frac{DB}{DC}$, in other words, for any point Y taken on Γ we have $\frac{YB}{YC} = \frac{DB}{DC}$. It means YD is the internal bisector of $\angle BYC$ and YD' is the external one.

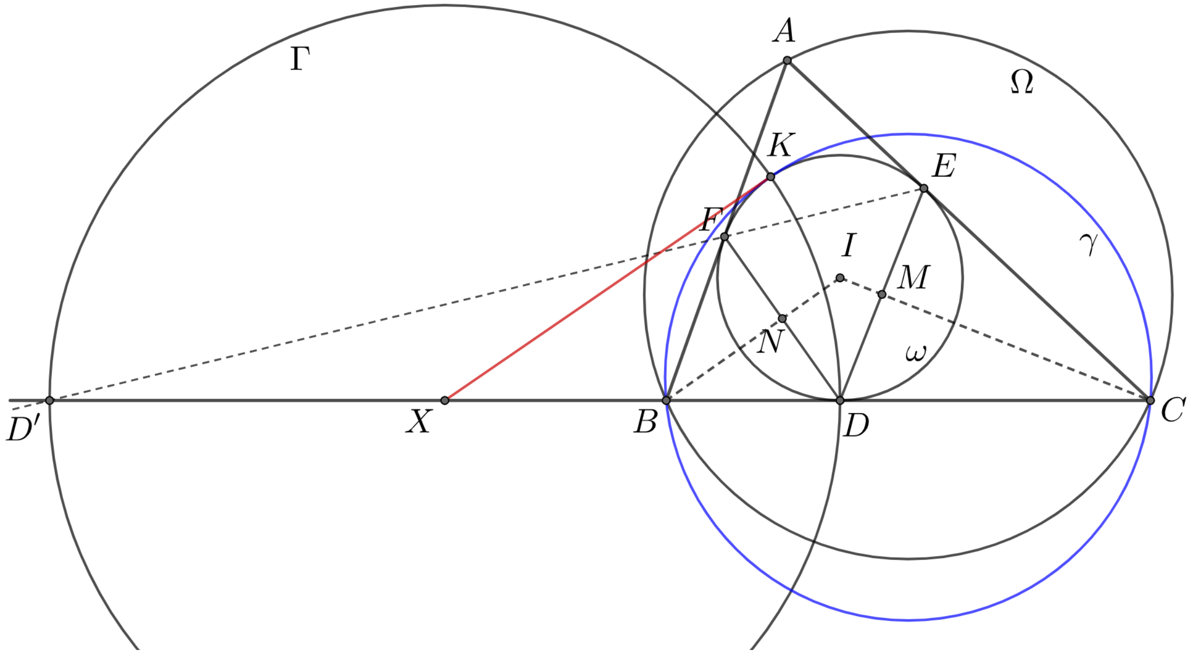


FIGURE 1. *
Figure 1

So Γ is a circle centred at X with radius XD. Let Γ meets with ω at points K and D ($K \neq D$). Then we get that KD and KD' are the internal and external bisectors of the triangle BKC, moreover XK is tangent to (BKC) (circumcircle of BKC, let's call it γ). On the other hand, $XK = XD$ and K lies on ω, so XK should be tangent to ω as well. Hence γ and ω are tangent to each other at point K.

So we defined the point K, and we claim the following.

Claim. The circumcircles of triangles BDM and CDN pass through the point K. (see *Figure 2*)

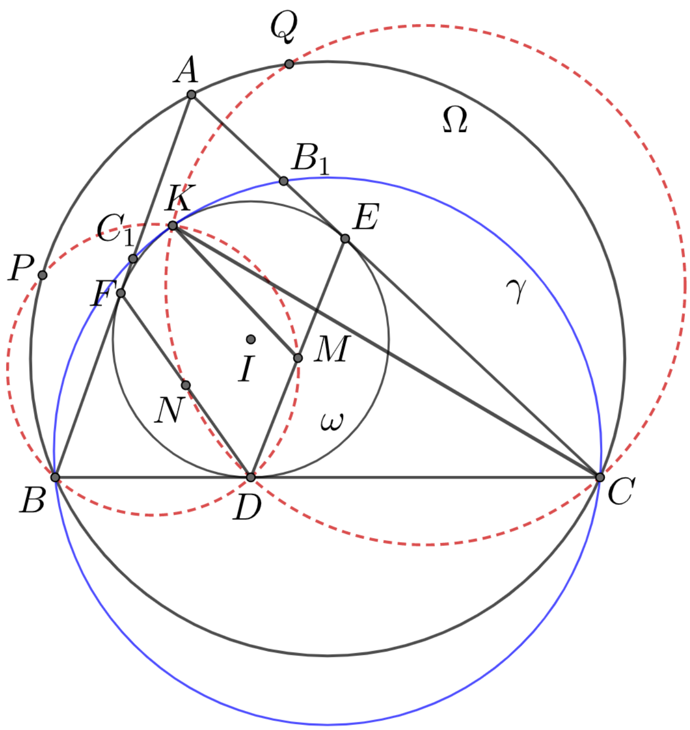


FIGURE 2. *
Figure 2

Proof. We only show that the circle (BDM) passes through K, and showing the other circle also passes through K follows in a similar way.

Let $\gamma \cap AC = B_1$ and we get a triangle BB_1C inscribed in γ which has mixtilinear incircle ω. In the triangle DKE, it is known that KC is the symmedian-line, KM is the median-line, and therefore, $\angle DKM = \angle CKE$. Moreover, we know that KD is the angle bisector of BKC and KE is the angle bisector of $\angle CKB_1$, so

$$\angle BKM = \angle BKD + \angle MKD = \frac{\angle BKC}{2} + \frac{\angle CKB_1}{2} = \frac{\angle BKB_1}{2} = \angle CDM$$

and $BKMD$ is cyclic. This proves our claim. □

Following the above result, we can obtain that BP, CQ and DK are concurrent (let's say at point R, see *Figure 3*) as the radical axes of circles Ω, (BDM), (CDN) are concurrent. From this, one can easily see that

$$\mathrm{Pow}_R(\Omega) = RP \cdot RB = RK \cdot RD = \mathrm{Pow}_R(\omega)$$

and R lies on ℓ, which is the radical axis of ω and Ω. So we may recognise R as the intersection point of the lines ℓ and KD.

On the other hand, we may return to the point X we defined earlier, since $XK^2 = XD^2 = XB \cdot XC$ and we get $\mathrm{Pow}_X(\omega) = \mathrm{Pow}_X(\gamma) = \mathrm{Pow}_X(\Omega)$, i.e. X lies on ℓ as well. From this, we can deduce more: if we consider points Y_1 and Y_2 as the intersections of Γ and Ω (see *Figure 3*), then XY_1 and XY_2 are tangent to Ω. Using

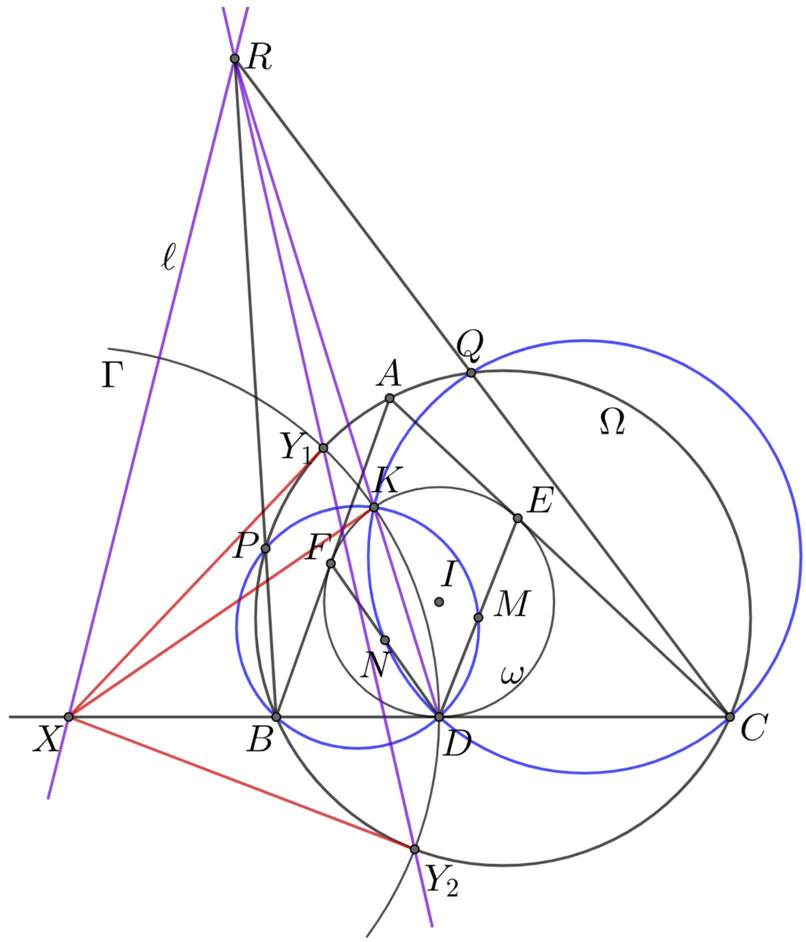


FIGURE 3. *
Figure 3

the radical axes theorem for the circles Γ, ω and Ω, we obtain that their radical axes are concurrent, i.e. Y_1Y_2, ℓ and KD are concurrent.

Since ℓ and KD meet at point R, we get that Y_1Y_2 passes through R. That means R lies on the polar line of X w.r.t. Ω, and therefore, X should lie on the polar line of R w.r.t. Ω. Hence X is the intersection of PQ and BC, i.e. PQ passes through X.

It is easy to see that NM passes through X as X being the midpoint of DD'. Furthermore, one may find that $BNMC$ is cyclic since $ID^2 = IN \cdot IB = IM \cdot IC$. So

$$XP \cdot XQ = XB \cdot XC = XN \cdot XM$$

and $PQMN$ is cyclic. This completes our solution. □

Remark 1. The proof of the claim can be omitted as it can be seen as a basic property comes together with mixtilinear incircles. The point M be the incenter of the triangle BB_1C.

Remark 2. The point K has some other properties as well. For instance, KD passes through the excenter I_a (the center of A-excircle) and the midpoint of the altitude AA_1 of the triangle ABC.

Remark 3. The statement of the problem can be changed to showing that PQ, MN, BC are concurrent as it is actually equivalent to saying that $PQMN$ is cyclic.

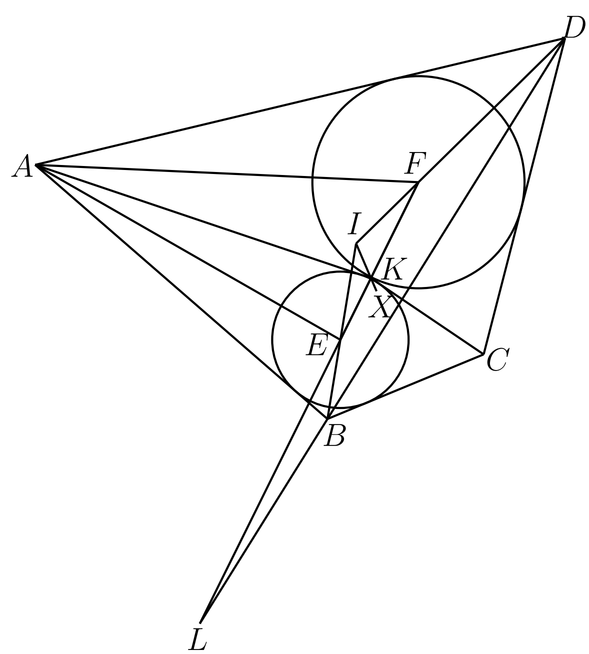
Remark 4. If we define the point $BQ \cap CP = Z(.)$, then O, I, Z are collinear. Both OZ and IZ are perpendicular to ℓ.

Alternative solution. Let X be the second intersection of PM with the circumcircle, Y similarly defined intersecting QN with circumcircle. Thanks to Reim's Theorem, CX is parallel to DE then X is midpoint of arc ACB. Similarly, Y is midpoint of arc ABC. Then XY is parallel to EF, parallel to MN. Since $PQXY$ is cyclic, $PQMN$ is cyclic thanks to Reim's Theorem.

Problem G5. Suppose a circle centered at I is inscribed in a convex quadrilateral $ABCD$. Let E and F be respectively on BI and DI such that $\angle EAF = \frac{1}{2}\angle BAD$. Let X be the intersection of DE and BF, and let K be the intersection of IX and EF. Prove that EF bisects the angle AKC.

Proposed by *Amedeo Bianchi & Leonardo Franchi, Italy*

Solution.



Let L be the intersection (possibly at infinity) of BD and EF. Let ω_1 and ω_2, respectively, be the circle with center E and tangent to AB and BC and the circle with center F and tangent to AD and DC. Let ω be the inscribed circle of $ABCD$. By Monge's theorem on ω, ω_1 and ω_2, B and D are collinear with the exsimilcenter of ω_1 and ω_2, which lies on EF and therefore is L.

Since, by a well-known application of Ceva's theorem and Menelaus's theorem on IEF, we have $\frac{LE}{LF} = \frac{KE}{KF}$, K is the insimilcenter of ω_1 and ω_2.

Since $\widehat{EAF} = \frac{1}{2}\widehat{BAD}$, the reflection of AB in AE (which is tangent to ω_1) and the reflection of AD in AF (which is tangent to ω_2) coincide. This line, which is one of the common internal tangents of ω_1 and ω_2, must pass through K.

We now claim that $\widehat{ECF} = \frac{1}{2}\widehat{BCD}$. Note that, if this claim is true, we have, similarly to before, that CK is a common internal tangent of ω_1 and ω_2, thus AK and CK are symmetrical with respect to EF, which is the thesis of the problem.

It now only remains to prove this lemma. We offer two approaches.

Approach 1. Let $\theta = \widehat{BAE} = \widehat{IAF}$ and $\phi = \widehat{EAI} = \widehat{FAD}$. Note that, by sine theorem on triangles ABE and AEI, we have $\frac{BE}{EI} = \frac{AB}{AI} \cdot \frac{\sin\theta}{\sin\phi}$ and, similarly, $\frac{DF}{FI} = \frac{AD}{AI} \cdot \frac{\sin\phi}{\sin\theta}$, yielding

$$\frac{BE \cdot DF}{EI \cdot FI} = \frac{AB \cdot AD}{AI^2}$$

If we define F' as the point on DI such that $\widehat{ECF} = \frac{1}{2}\widehat{BCD}$, we similarly have $\frac{BE \cdot DF'}{EI \cdot F'I} = \frac{CB \cdot CD}{CI^2}$. We wish two show that $F = F'$, that is, $\frac{DF}{FI} = \frac{DF'}{F'I}$, which is equivalent to $\frac{AB \cdot AD}{AI^2} = \frac{CB \cdot CD}{CI^2}$.

We ultimately have

$$\frac{AB \cdot AD}{AI^2} = \frac{\sin\widehat{AIB} \cdot \sin\widehat{AID}}{\sin\frac{\widehat{B}}{2} \cdot \sin\frac{\widehat{D}}{2}} = \frac{\sin\widehat{CIB} \cdot \sin\widehat{CID}}{\sin\frac{\widehat{B}}{2} \cdot \sin\frac{\widehat{D}}{2}} = \frac{CB \cdot CD}{CI^2}$$

because $\widehat{AIB} + \widehat{CID} = \widehat{CIB} + \widehat{AID} = 180°$.

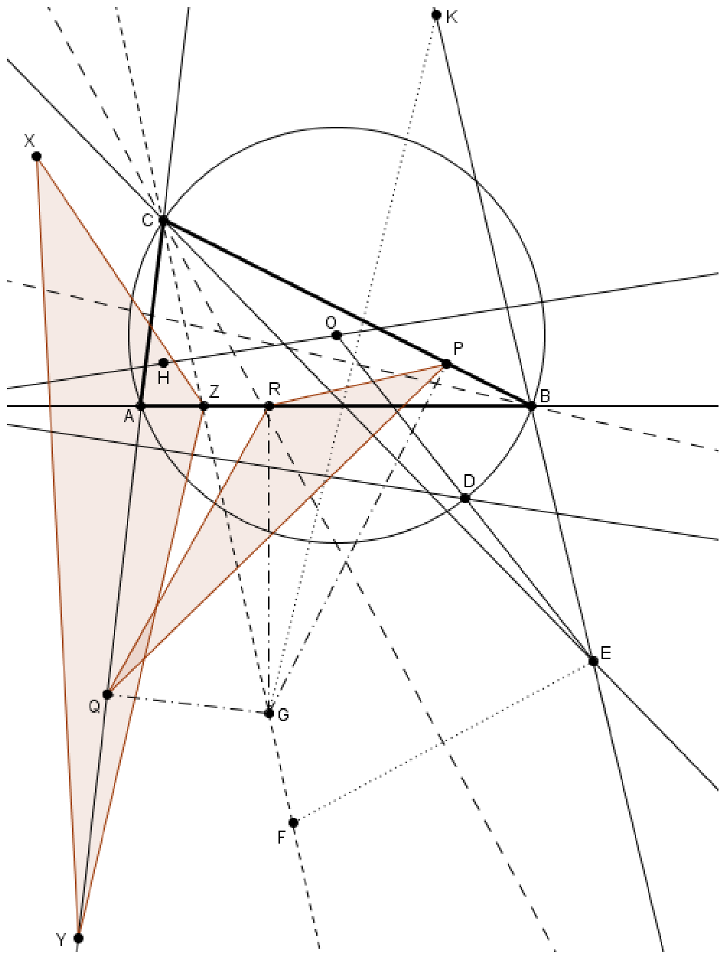
Approach 2. We recall that, in a quadrilateral $ABCD$, an internal point P has an isogonal conjugate if and only if $\widehat{APB} + \widehat{CPD} = 180°$. Thus I has an isogonal conjugate in quadrilateral $ABCF$, which must lie on BI and AE (because $\widehat{EAF} = \frac{1}{2}\widehat{BAD}$), thus it is E, therefore $\widehat{ECF} = \frac{1}{2}\widehat{BCD}$.

Remark. A way to rephrase the definiton of K is to define L in the statement of the problem and K on EF such that $\frac{LE}{LF} = \frac{KE}{KF}$. This avoids recalling the well-known lemma with Ceva and Menelaus. □

Problem G6. Let ABC be a triangle with circumcenter O and othocenter H. Let ℓ be the line through O and H. Let D be a point on the circumcircle of the triangle ABC such that the reflection of D across the line AB is a point on ℓ that is different from H. Let E be a point such that D is the midpoint of OE. Let F be the reflection of E across the bisector of $\angle ACB$. Let G be an arbitrary point on the line CF, and let K be the reflection of G across the bisector of $\angle ABC$. Let P, Q, R be the foots of perpendiculars from the point G to the lines BC, CA, AB respectively. Let X, Y, Z be the intersections of lines GA, GB, GC with the lines BC, CA, AB respectively. Suppose that $\angle PQR = \angle XYZ$. Prove that B, E and K are collinear.

Proposed by *Bojan Basic, Armenia*

Solution.



We first prove that, from the fact that G is on the line CF and $\measuredangle PQR \cong \measuredangle XYZ$, we can conclude $\triangle PQR \sim \triangle XYZ$.

Denote by G' the image of point G under inversion with respect to the circumcircle of $\triangle ABC$. Then $\triangle PQR \sim \triangle XYZ$ if and only if circumcircles of $\triangle AYZ$, $\triangle XBZ$ and $\triangle XYC$ intersect in one point, in fact the point G' (which is a well-known characterization of Miquel point). Denote by G^* isogonal conjugate of the point G. We shall further need the point U: intersection of tangents to the circumcircle of $\triangle ABC$ in B and C, as well as V: reflection of O with respect to the point A. The following assertion is the key one: the points A, Y, Z and G' are concyclic if and only if the point G^* lies on the circle γ_A, defined as the circle over the diameter UV. Let us prove that assertion.

We shall work with barycentric coordinates. Let us first find the equation of the circle γ_A. It passes through the points $\overline{B}$ and $\overline{C}$, that are reflections of B and C with respect to the lines CA and AB, respectively, and can be defined as the locus of points M such that $\measuredangle \overline{B}M\overline{C} = -2\measuredangle ABC$ (here the angles are oriented). Its equation is

$$2e_a(a^2yz + b^2zx + c^2xy) + (b^2c^2x + 2c^2e_cy + 2b^2e_bz)(x + y + z) = 0,$$

where $e_a = bc\cos\measuredangle A$ and similarly for e_b and e_c (and, of course, a, b and c are side lengths). Let now $p : q : r$ be the barycentric coordinates of the point G. Then the circumcircle k_A of the $\triangle AYZ$ has equation

$$a^2yz + b^2zx + c^2xy - p(x+y+z)\left(\frac{c^2}{p+q}y + \frac{b^2}{p+r}z\right) = 0$$

and its inverse image, say k'_A, with respect to the circumcircle of $\triangle ABC$, has equation

$$(a^2(p^2-qr)+(b^2-c^2)p(q-r))(a^2yz+b^2zx+c^2xy)-pa^2(x+y+z)(c^2(p+r)y+b^2(p+q)z) = 0.$$

Now, since $G' \in k_A$, we have $G \in k'_A$. Replacing p, q, r in the above equation by x, y, z gives the locus of points G such that $G' \in k_A$. Then, replacing x, y, z by $(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z})$ gives the locus of points G^* such that $G' \in k_A$; however, we see that the equation obtained this way is precisely the equation for γ_A obtained above. This proves the assetion.

In other words, the proven assertion reduces to the fact that γ_A, γ_B and γ_C (where γ_B and γ_C are defined analogously to γ_A) intersect in one point, in fact the point G^*. However, it is not hard to show that the point E lies on all the circles γ_A, γ_B i γ_C (the fastest way, though, might be by calculating, e.g. by complex numbers). Therefore, considering everything so far, we have that the condition $\triangle PQR \sim \triangle XYZ$ is equivalent to the assertion that the points G and E are isogonal conjugates of each other. However, since the lines CF and CE are reflections of each other with respect to the bisector of $\measuredangle C$, and since we have $\measuredangle PQR \cong \measuredangle XYZ$ and there could be at most one point on the line CF having this property, it follows that G is precisely the point for which we have $\triangle PQR \sim \triangle XYZ$.

It is easy to finish the problem from here. Namely, since we got earlier that the considered similarity is equivalent to the condition that the points G and E are isogonal conjugates of each other, this means that the point G lies on the reflection of the line BE with respect to the bisector of $\measuredangle B$. However, this is only the reformulation of the assertion that the points B, E and K are collinear, which was to be proved. $\square$

10. Solutions-Number Theory

Problem N1. Find all integer values of a for which $X^2 + X + a$ divides $X^{13} + X - 90$.

Proposed by *Dinu Şerbaňescu, Romania*

Solution. Write $f = X^{13} + X - 90 = (X^2 + X + a)P(X)$ and notice that $P(X)$ is a polinomial with integer coefficients. Set $x = 0$ to get that a divides 90, and set $x = -1$ to obtain that a divides 92, therefore $a \in \{-2, -1, 1, 2\}$. We claim that $\boxed{a = 2}$.

First, we eliminate the cases when $a = -2, 1, -1$:

If $a = -2$, then $X^2 + X - 2 = (X - 1)(X + 2)$, implying $X - 1 \mid f$, which is false, as $f(1) = -88 \neq 0$.

Suppose $a = 1$ and set $x = 2$ to get $f(-2) = -2^{13} - 92 = 3P(-2)$. Notice that 3 divides $(2^{13} + 1) + 90$ to reach again a contradiction.

Finally, if $a = -1$ one notice that $f(-4) = -4^{13} - 94 = 11P(-4)$, implying $11 \mid 4^{13} + 6$. On the other hand, $4^{10} \equiv 4^3 \pmod 1 1$, hence $4^{13} + 6 \equiv 4^3 + 6 = 70 \pmod 1 1$, a contradiction.

To conclude, write $X^{13} + X - 90 = (X^2 + X + 2)P(X)$, where $P(X) =$
$X^{11} - X^{10} - X^9 + 3X^8 - X^7 - 5X^6 + 7X^5 + 3X^4 - 17X^3 + 11X^2 + 23X - 45$.

Alternatively, show that any root z of $g = X^2 + X + 2$ is also a root of the polynomial f:

$$
\begin{aligned}
z^2 &= -z - 2 \Rightarrow \\
z^4 &= z^2 + 4z + 4 = -z - 2 + 4z + 4 = 3z + 2 \Rightarrow \\
z^8 &= 9z^2 + 12z + 4 = -9z - 18 + 12z + 4 = 3z - 14 \Rightarrow \\
z^{12} &= z^8 z^4 = (3z + 2)(3z - 14) = 9z^2 - 36z - 28 = -45z - 46 \Rightarrow \\
z^{13} &= -45z^2 - 46z = 45z + 90 - 46z = -z + 90 \Rightarrow \\
f(z) &= z^{13} + z - 90 = 0.
\end{aligned}
$$

□

Problem N2. For $n > 1$, define the function $C(n)$ as follows:

$$C(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 2), \text{ where } n = \prod_{i=1}^{k} p_i^{\alpha_i}, p_i \text{ prime}$$

Prove that if

$$C(n) | \phi(n) - 1,$$

then n is an odd prime or n has at least 7 prime distinct divisors.

Proposed by *Tudor Popescu, United States*

Solution. Let $MC(n) = \phi(n) - 1$. Obviously, if $M = 1$, then n must be prime and the condition holds. Assume now that n is composite and $M > 1$. Since $\phi(n)$

is even, we must have that M is odd, so $M \geq 3$. Moreover, n is odd, as otherwise $C(n) = 0$. Furthermore, n must be square-free, since if $p^2|n$, then $p|C(n), \phi(n)$, so p divides $MC(n) = \phi(n) - 1$. But then $p|\phi(n) - (\phi(n) - 1) = 1$, which is false.
Obviously, if $M = 1$, then n must be prime and the condition holds. Assume now that n is composite and $M > 1$. Since $\phi(n)$ is even, we must have that M is odd, so $M \geq 3$. First consider the case when $n = p_1p_2, p_i$ prime. Then

$$M(p_1 - 2)(p_2 - 2) = (p_1 - 1)(p_2 - 1) - 1 \Rightarrow M = 1 + \frac{1}{p_1 - 2} + \frac{1}{p_2 - 2}$$

Therefore, $M \leq 1 + \frac{1}{1} + \frac{1}{3} < 3$, contradicting the fact that $M \geq 3$.
If n has less than 7 divisors, we prove that $3 \mid n$. Assume that $3 \nmid n$ and $k \leq 6$, where k is the number of distinct prime divisors of n. Then

$$M < \frac{\phi(n)}{C(n)} = \prod_{i=1}^{k} \frac{p_i - 1}{p_i - 2} \leq \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \frac{16}{15} \cdot \frac{18}{17} < 3$$

which is false. Therefore, $3|n$, and $n = 3p_2 \dots p_k$. If $k \leq 6$, then

$$M < \frac{\phi(n)}{C(n)} = \frac{2}{1} \prod_{i=2}^{k} \frac{p_i - 1}{p_i - 2} \leq \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \frac{16}{15} < 5$$

Therefore, $M = 3$, so we have that $3C(n) = \phi(n) - 1$, hence

$$3(p_2 - 2) \dots (p_k - 2) = 2(p_2 - 1) \dots (p_k - 1) - 1$$

But this equation obviously has no solutions modulo 3, as $p_i - 1 \equiv 0, 1 \pmod 3$. Therefore, n must have at least 7 prime divisors, as desired. □

Problem N3. Determine all positive primes p such that there exists integers x and y such that x and y are not divisible by p and $x^2 + y^3 + 1$ is divisible by p.

Proposed by *Tudor Popescu, United States*

Solution. All the primes p except 2 and 7.
For $p = 2$, $x^2 + y^3 + 1 \equiv 1 + 1 + 1 \equiv 1 \pmod 2$. For $p \geq 3$ we will solve using 2 cases. If $p \equiv 1 \pmod 4$.
We have $\left(\frac{-9}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{9}{p}\right) = 1 \cdot 1 = 1$ and we can find x such that $x^2 \equiv -9 \pmod p$. Taking this x and $y = 2$ we have $x^2 + y^3 + 1 \equiv -9 + 2^3 + 1 \equiv 0 \pmod p$.

If $p \equiv 3 \pmod 4$.
For $p = 3$ take $x = y = 1$ and $x^2 + y^3 + 1 \equiv 0 \pmod 3$.
For $p = 7$ we know that $x^2 \equiv 1, 2$ or $4 \pmod 7$, $y^3 \equiv 1$ or $6 \pmod 7$, $x^2 + y^3 \not\equiv 0 \pmod 7$ and $x^2 + y^3 + 1$ can not be divisible by 7.
For $p > 7$, note that $\left(\frac{-28}{p}\right) = \left(\frac{4}{p}\right)\left(\frac{-1}{p}\right)\left(\frac{7}{p}\right) = -\left(\frac{7}{p}\right)$. If $(7/p) = 1$, then there exists x such that $x^2 \equiv 7 \pmod p$ and taking $y = -2$ we have $x^2 + y^3 + 1 \equiv 7 + (-2)^3 + 1 \equiv 0 \pmod p$. If $\left(\frac{7}{p}\right) = -1$, then $\left(\frac{-28}{p}\right) = 1$ there exists x such that $x^2 \equiv -28 \pmod p$ and using $y = 3$ we get the congruence $x^2 + y^3 + 1 \equiv -28 + 3^3 + 1 \equiv 0 \pmod p$. □

Problem N4. Let a be an odd number. Define the sequence $\{x_n\}_{n\geq 0}$ by $x_0 = 1, x_1 = a$, and $x_{n+2} = 2ax_{n+1} - x_n$. Show that for every prime power p^l, there is a choice of sign for which

$$x_{p^l} \pm x_{p^{l-1}} \equiv 0 \pmod{p^l}.$$

Proposed by *Joao Campos-Vargas, United States*

Solution.

Write $a^2 - 1 = db^2$ with d square-free. We can derive

$$x_n = \frac{\gamma^n + \gamma^{-n}}{2}$$

where $\gamma = a + b\sqrt{d}$. We now consider two cases:

(i) $p \mid a$: Such p must be odd. Notice that $p \mid x_1$. We show that

$$v_p(x_{pn}) = v_p(x_n) + 1$$

whenever $p \mid x_n$. This will give us $p^l \mid x_{p^l} \pm x_{p^{l-1}}$ for any choice of sign.

To prove this, write $\gamma^n + \gamma^{-n} = p^\alpha T$ with $\alpha = v_p(x_n)$ and T not divisible by p (notice we don't have to worry with the factor 2 in the denominator of x_n because p is odd). Then,

$$\gamma^n = -\gamma^{-n} + p^\alpha T.$$

Taking both sides to the power p, we obtain

$$\gamma^{np} = -\gamma^{-np} + p^{\alpha+1}T'$$

and one can check that T' is an integer not divisible by p by using the binomial expansion, or alternatively considering the finite field that contains $\sqrt{d}$. The result follows in this case.

(ii) $p \not| a$: The strategy here is to notice that $x_{p^{l-1}}$ divides x_{p^l} and to show that any prime factor dividing the ratio $x_{p^l}/x_{p^{l-1}}$ is congruent to ± 1 modulo p^l. The result follows from this.

For the first part, observe that

$$\frac{\gamma^{p^l} + \gamma^{-p^l}}{\gamma^{p^{l-1}} + \gamma^{-p^{l-1}}} = \sum_{n=-\frac{p-1}{2}}^{\frac{p-1}{2}} \gamma^{2p^{l-1}n}$$

is an integer. For the second part, we first note that p does not divide x_{p^l}. To see that, consider the cases:

- $\left(\frac{d}{p}\right) = 0, 1$. In this case $(a + b\sqrt{d})^p \equiv_p a + b\sqrt{d}$. Hence $x_{p^l} \equiv a \pmod{p}$ is not divisible by p.
- $\left(\frac{d}{p}\right) = -1$. In this case $(a+b\sqrt{d})^p \equiv_p a - b\sqrt{d}$. We still get $x_{p^l} \equiv a \pmod{d}$ not divibible by p.

Now take any prime factor r of $x_{p^l}/x_{p^{l-1}}$. We know $r \neq p$. We must have that r does not divide $x^{p^{l-1}}$, because by the argument above we know that we would gain no new r factors in x_{p^l} besides those in $x_{p^{l-1}}$. Now, we know that

$$\gamma^{p^l} + \gamma^{-p^l} = 0$$

and the equality takes place modulo p. Working over a finite field extension that contains $\sqrt{d}$, that is, either $\mathbb{F}_r$ or $\mathbb{F}_{r^2}$ we get that the order of γ divides $4p^l$. On the other hand, it must be a multiple of p^l otherwise r would divide $x_{p^{l-1}}$. It follows that p^l divides the size of the multiplicative group of the finite field. In this case, p^l divides $r^2 - 1$ hence $r \equiv \pm 1 \pmod{p^l}$. This concludes the problem.

Problem N5. Let n be a positive integer. Prove there exists an N such that, for every prime $p > N$ there exist n consecutive numbers such that all of them are quadratic residues.

Folklore, suggested by *Andrei Bud, Germany*

Solution. We will use **Van der Waerden's Theorem**. We color the quadratic residues in red and the other numbers in blue. Choosing $p > W(2, n^2 + n)$, the theorem implies we have a monochromatic arithmetic progression of length $n^2 + n$. Let it be

$$a, a + d, a + 2d, \ldots, a + (n^2 + n - 1)d$$

We consider the integer d' satisfying $d' \equiv d^{-1} \mod p$. Then the sequence

$$ad', ad' + 1, ad' + 2, \ldots, ad' + (n^2 + n - 1)$$

is a monochromatic arithmetic progression. If the color is red, there is nothing to prove. Assume the color is blue (i.e. no number in the sequence is a quadratic residue).

We also look at the numbers $1, 2, \ldots, n$. If all of them are quadratic residues, there is nothing left to prove. We can assume that at least one of them, denoted s is not a quadratic residue.

Because $s \leq n$, the sequence

$$ad', ad' + 1, ad' + 2, \ldots, ad' + (n^2 + n - 1)$$

contains at least n consecutive multiples of s, denoted

$$r(m + 1), r(m + 2), \ldots, r(m + n)$$

Because r^{-1} is not a quadratic residue, it follows that

$$m + 1, m + 2, \ldots, m + n$$

are all quadratic residues, hence the problem is solved. □

Problem N6. For a positive integer n, the symbol $!n$ is defined in the following way:

$$a!n = \prod_{\substack{1 \leqslant i \leqslant a \\ i \equiv a \pmod{n}}} i.$$

Find all positive integers n, where $n \geqslant 2023$, which satisfy the following: there exist infinitely many positive integers k such that, for some positive integers $a_1, a_2, \dots, a_k$ for which $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, $a_k - a_1 > n$, and which are all congruent modulo n, and some positive integer t, we have

$$a_1!n \cdot a_2!n \cdots a_k!n = t^k.$$

Note. The problem can be proposed even without the constraint $n \geqslant 2023$, but this adds some casework to the solution.

Proposed by *Bojan Basic, Armenia*

Solution. We claim that for each positive integer n, $n \geqslant 2023$, there exist infinitely many such numbers k.

We first observe that, for a given n, it is enough to find one such k, as if k has the considered property, than all the numbers kl for $l \in \mathbb{N}$ also have that property. Indeed, if $a_1, a_2, \dots, a_k$ are the corresponding positive integers, then for the numbers $a_1, a_1, \dots, a_1, a_2, a_2, \dots, a_2, \dots, a_k, a_k, \dots, a_k$, where each of the numbers is repeated l times, we have

$$a_1!_{(n)} \cdot a_1!_{(n)} \cdots a_1!_{(n)} \cdot a_2!_{(n)} \cdot a_2!_{(n)} \cdots a_2!_{(n)} \cdots a_k!_{(n)} \cdot a_k!_{(n)} \cdots a_k!_{(n)} = (a_1!_{(n)} \cdot a_2!_{(n)} \cdots a_k!_{(n)})^l = (t^k)^l = t^{kl}.$$

Let us now construct one such integer k for each n. The basic idea is this: if there are two perfect powers whose difference equals n and that are both greater than n, then k can be constructed in the following way. Let $d^q - c^p = n$. Let u be the least positive integer such that $uq \geqslant p$ (clearly, $u < p$, since $(p-1)q - p = (p-1)(q-1) - 1 \geqslant 1 - 1 = 0$). Then:

$$((c^p - n)!_{(n)})^{pq-uq} (c^p!_{(n)})^{uq-p} (d^q!_{(n)})^p = ((c^p - n)!_{(n)})^{pq} (c^p)^{uq} (d^q)^p = ((c^p - n)!_{(n)} c^u d)^{pq},$$

that is, we may choose $k = pq$.

For example, for $n = 1$ (although this case is unnecessary), as $3^2 - 2^3 = 1$, the described construction gives that $(7!)^3 \cdot 8! \cdot (9!)^2$ is a perfect 6^{th} power. The described construction also resolves all positive integers n that can be presented as a difference of two squares, both greater than n. If n is odd, then, writing $n = d^2 - c^2 = (d-c)(d+c)$ and choosing $d - c = 1$ and $d + c = n$, we get $n = (\frac{n+1}{2})^2 - (\frac{n-1}{2})^2$; therefore, this resolves all odd positive integers n such that $(\frac{n-1}{2})^2 > n$, that is, $n^2 - 6n + 1 > 0$, which reduces to $n \geqslant 7$ (then our construction gives that $(((\frac{n-1}{2})^2 - n)!_{(n)})^2 ((\frac{n+1}{2})^2!_{(n)})^2$ is a perfect 4^{th} power). If n is divisible by 4, from $n = d^2 - c^2 = (d-c)(d+c)$, choosing $d - c = 2$ and $d + c = \frac{n}{2}$, we get $n = (\frac{n}{4}+1)^2 - (\frac{n}{4}-1)^2$; therefore, this resolves all the numbers n divisible by 4 for which $(\frac{n}{4}-1)^2 > n$, that is, $n^2 - 24n + 16 > 0$, which reduces to $n \geqslant 24$. Beside that, the presented construction for odd $n \geqslant 7$ can be simply generalized to the case when n has an odd divisor greater than 5: namely, for $n = zn'$, z odd and $z \geqslant 7$, we have

$$\begin{aligned}
&\left(\left(\left(\frac{z-1}{2}\right)^2 n' - zn'\right)!_{(n)}\right)^2 \left(\left(\left(\frac{z+1}{2}\right)^2 n'\right)!_{(n)}\right)^2 \\
&= \left(\left(\left(\frac{z-1}{2}\right)^2 n' - zn'\right)!_{(n)}\right)^4 \left(\left(\frac{z-1}{2}\right)^2 n'\right)^2 \left(\left(\frac{z+1}{2}\right)^2 n'\right)^2 \\
&= \left(\left(\left(\frac{z-1}{2}\right)^2 n' - zn'\right)!_{(n)}\right)^4 \left(\frac{z-1}{2}\right)^4 \left(\frac{z+1}{2}\right)^4 (n')^4.
\end{aligned}$$

Altogether, this resolves all the numbers n greater than 20 and divisible by 4, as well as all the numbers n that have an odd divisor greater than 5. In particular, all the numbers $\geqslant 2023$ are resolved, which finishes the problem.

Note. Actually, with a little more work, it can be shown that the constraint $n \geqslant 2023$ is unnecessary, as the problem statement is valid for each positive integer n. Let us show the proof for the remaining cases: namely, when n is a power of 2 not greater than 16, when n is of the form $2^\alpha \cdot 3$ for $\alpha \leqslant 2$ (that is, $n = 3, 6, 12$), and when n is of the form $2^\alpha \cdot 5$ for $\alpha \leqslant 2$ (that is, $n = 5, 10, 20$). We first show a contraction that resolves all the powers of 2 (even arbitrarily large). Let $n = 2^\alpha$. The construction will actually be a direct generalization of the presented construction for $n = 1$. Namely, we consider expressions of the form

$$((2^\alpha\cdot 7)!_{(n)})^{k-u-v}((2^\alpha\cdot 8)!_{(n)})^u((2^\alpha\cdot 9)!_{(n)})^v = ((2^\alpha\cdot 7)!_{(n)})^k(2^\alpha\cdot 8)^{u+v}(2^\alpha\cdot 9)^v = ((2^\alpha\cdot 7)!_{(n)})^k 2^{(\alpha+3)(u+v)+\alpha v}3^{2v}.$$

It is, therefore, enough to find u and v such that $2^{(\alpha+3)(u+v)+\alpha v}$ and 3^{2v} are perfect k^{th} powers for some k, $k > u + v$. Let $u = 1$. Then we need to choose v such that $(\alpha + 3)(v + 1) + \alpha v$ and $2v$ have a common divisor greater than $v + 1$. It is not hard to see that for $v = \alpha + 3$ the first expression reduces to $(\alpha+3)(2\alpha+4)$, which is divisible by $2(\alpha+3)$, and this is precisely the second expression; therefore, for the parameters $u = 1$, $v = \alpha + 3$ and $k = 2(\alpha + 3)$ we obtain what was needed.

Finally, the six remaining values of n shall be considered on the case-by-case basis. Since $128 - 125 = 3$ and $32 - 27 = 5$, this resolves the cases $n = 3$ and $n = 5$ (because there are perfect powers on the left-hand side). For $n = 6$ we have $6!_{(6)} \cdot 6!_{(6)} \cdot 18!_{(6)} = 6^3 \cdot 12 \cdot 18 = (36)^3$, while for $n = 12$ we have $24!_{(12)} \cdot 24!_{(12)} \cdot 48!_{(12)} = (12 \cdot 24)^3 \cdot 36 \cdot 48 = (12 \cdot 24 \cdot 12)^3$. For $n = 10$ we have $80!_{(10)} \cdot 80!_{(10)} \cdot 90!_{(10)} \cdot 100!_{(10)} = (80!_{(10)})^4 \cdot 90^2 \cdot 100 = (80!_{(10)} \cdot 3 \cdot 10)^4$, while for $n = 20$ we have $60!_{(20)} \cdot 60!_{(20)} \cdot 100!_{(20)} = (60!_{(20)})^3 \cdot 80 \cdot 100 = (60!_{(20)} \cdot 2 \cdot 10)^3$. This completes the solution. □

N7. Find all polynomials $P(x)$ with integer coefficients such that for all positive integers m, n we have:

$$m + n \mid P^{(m)}(n) - P^{(n)}(m).$$

Proposed by *Navid Safaei, Iran*

Solution. Assume that $P(x)$ be a non-constant polynomial. Moreover, take n large enough such that $P(n) \neq 0$. Consider the sequence $a_m = P^{(m)}(n)$. Let $m = q - n$ it follows that q divides $P^{(m)}(n) - P^{(n)}(-n)$. The sequence a_m is eventually periodic mod q. Let T be the length of its fundamental period, we can assume that $T \leq q$. If $T = q$ then the set $\{P^{(r)}(n), \ldots, P^{(r+q)}(n)\}$ forms the complete residue system mad q. If $T < q$ then $\gcd(T, q) = 1$. Let s be an arbitrary integer in $\{1, \ldots, q\}$ it follows that there is a positive integer k such that $m + qk \equiv s (\text{mod} T)$. Setting $(m, n) = (m + qk, n)$ it follows that q divides

$$P^{(m+qk)}(n) - P^{(n)}(m+qk) \equiv P^{(s)}(n) - P^{(n)}(m).$$

Hence, the sequence must be 1 -periodic. That is,

$$P(a_i) \equiv a_i (\bmod q) \text{ and } a_i \equiv a_{i+1} (\bmod q).$$

Therefore $T \in \{1, q\}$.

Note that if for all large enough q, we have $T = q$ then it follows that for all large enough q the set $\{P(0), \ldots, P(q-1)\}$ is complete residue modulo q. If $P(x)$ is not linear, consider $P(x+1) - P(x)$ and use *Schur*'s theorem to arrive at contradiction.

Thus, there are infinitely many q with $T = 1$. It follows that there is a positive integer t such that $P(t) \equiv t (\bmod q)$. In particular, $P^{(q-n+1)}(n) \equiv P^{(q-n)}(n) (\bmod q)$ far a fixed n and infinitely many q.

Further, since

$$P^{(q-n)}(n) - P^{(n)}(-n) \mid P(P^{(q-n)}(n)) - P(P^{(n)}(-n)) = P^{(q-n+1)}(n) - P^{(n+1)}(-n),$$

it follows that q divides $P^{(q-n+1)}(n) - P^{(n+1)}(-n)$. Since

$$P^{(q-n+1)}(n) \equiv P^{(q-n)}(n) (\bmod q),$$

it follows that

$$P^{(n+1)}(-n) \equiv P^{(n)}(-n) (\bmod q).$$

That is, for all large $l, l \geq n; P^{(l)}(-n)$ is a root of $P(x) \equiv x (\bmod q)$. Hence, choose $q > |P^{(n)}(-n)|, |P^{(n+1)}(-n)|$ it follows that

$$P^{(n+1)}(-n) = P^{(n)}(-n).$$

Hence, $P^{(n)}(-n)$ is a root of $P(x) - x$. choosing q large enough, it follows that $P(x) - x$ can have $P^{(n)}(-n), P^{(n+1)}(-n), \ldots$ as its root. Impossible. □