INTERNATIONAL MATHEMATICS SUMMER CAMP IMSC23

MOCK TEST 2-COMBINATORICS

Date: Tuesday, 22nd June 2023 **Time**: 13:10-17:40
Number of problems: 3 **Total points**: 21

Problems

Problem 1. An international conference consists of 2023 representatives from each of 2023 different countries. Prove that 2023^2 people can be seated around a large round table such that if A and B are two distinct representatives from the same country, then the people sitting to the immediate left of A and to the immediate left of B are from different countries.

Solution. We prove this by induction on n which is the number of countries and also the number of representatives from each country. Label the people with their nationality, from 1 up to n (so n people get the same label i for every $1 \leq i \leq n$). For $n = 1$ there is nothing to do. For $n = 2$ we can arrange them as 1122 where the last 2 is to the immediate left of the first 1. For $n \geq 3$, by induction, we prove that there is an arrangement starting with 1 and ending with 2.

Once one has a solution for $n-1$, construct a solution by adding $1n2n3n4\ldots n(n-1)nn$ on the left. Note that the numbers to the immediate left of n are all different. The numbers to the immediate left of i ($1 \leq i \leq n-1$) are also different because the only new number to the left of an i is n. Thus we have a desired construction for every n, including 2023. □

Problem 2. Two players Alex and Ben play the following game. At the start of the game, Alex chooses a positive integer n and n positive integers $x_1, x_2, \ldots, x_n$. Alex keeps $x_1, x_2, \ldots, x_n$ secret, and truthfully tells n and $x_1 + x_2 + \cdots + x_n$ to Ben. Ben now tries to obtain information about $x_1, \ldots, x_n$ by asking Alex questions as follows: each question consists of Ben specifying an arbitrary nonempty proper subset B of $\{1, 2, \ldots, n\}$, and asking Alex for a subset A of $\{1, 2, \ldots, n\}$ such that

$$A \neq B \text{ and } \sum_{i \in A} x_i = \sum_{j \in B} x_j.$$

After each question, if Alex fails to answer it truthfully, Ben wins the game immediately. After Ben asks as many questions as he wants, if Ben can specify the values of $x_1, x_2, \ldots, x_n$, then Ben wins the game; otherwise Ben loses. Prove that Ben can guarantee a win.

Solution. To win the game, Ben asks a total of $2^n - 2$ questions — each question specifies a distinct nonempty proper subset of $\{1, 2, \ldots, n\}$. Without loss of generality, Alex answers each question truthfully, and Ben obtains $2^n - 2$ linear equations. To find out $x_1, x_2, \dots, x_n$, Ben can naively enumerate all possible values of $x_1, x_2, \dots, x_n$ that sum up to $x_1 + x_2 + \dots x_n$, which Alex told Ben in advance, and check if each possibility satisfies all the linear equations.

It suffices to show that there is only one possibility that satisfies all the linear equations. Assume for the sake of contradiction that there are two distinct solutions, say $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$. Let $B = \{j : a_j \geq b_j\}$. Since $a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_n$, and the two sequences are different, clearly B is a nonempty proper subset of $\{1, 2, \dots, n\}$. Since B was specified as one of Ben's questions, Alex has responded with a subset A such that

$$A \neq B, \quad \sum_{i \in A} a_i = \sum_{j \in B} a_j, \quad \sum_{i \in A} b_i = \sum_{j \in B} b_j.$$

This implies that

$$\sum_{i \in A} a_i - b_i = \sum_{j \in B} a_j - b_j,$$

which implies that

$$\sum_{i \in A \setminus B} a_i - b_i = \sum_{j \in B \setminus A} a_j - b_j.$$

Because of our choice of B, the left hand side is strictly negative, while the right hand side is non-negative. This is a contradiction. $\square$

Problem 3. In the land of Heptanomisma, four different coins and three different banknotes are used, and their denominations are seven different natural numbers. The smallest denomination of a banknote is greater than the sum of the denominations of the four different coins. A tourist has exactly one coin of each denomination and exactly one banknote of each denomination, but he cannot afford the book on numismatics he wishes to buy. However, the mathematically inclined shopkeeper offers to sell the book to the tourist at a price of his choosing, provided that he can pay this price in more than one way. (The tourist can pay a price in more than one way if there are two different subsets of his coins and notes, the denominations of which both add up to this price.) Find the natural number N such that (a) the tourist can purchase the book if the denomination of each banknote is smaller than N, and (b) the tourist may have to leave the shop empty-handed if the denomination of the largest banknote is N.

Solution. We will prove that $N = 49$.

Let the denominations of the coins and notes be $C_1 < C_2 < C_3 < C_4$ and $N_1 < N_2 < N_3$, respectively. Define $C = C_1 + C_2 + C_3 + C_4$ to be the largest amount that can be payed with coins only. The condition of the problem is thus $C < N_1$.

(a) Suppose to the contrary that, whatever the price of the book, the tourist can pay for it in no more than one way. Consider the $3 \cdot 2^4 = 48$ hands of exactly two notes and any number of (or possibly no) coins. Each of these has one of the N_3 different values v with $N_1 + N_2 \leq v < N_1 + N_2 + N_3$, since $C < N_1$. Hence $48 \leq N_3 < 49$, so $N_3 = 48$. Next consider the $3 \cdot 2^4 = 48$ hands of exactly one note and any number of coins. By the above, these cannot have a value greater than or equal to $N_1 + N_2$, since these can be realised using two notes and some coins. Hence they must each have one of the N_2 different values v with $N_1 \leq v < N_1 + N_2$. This implies $48 \leq N_2 < N_3 = 48$, which is a contradiction.

(b) Consider the denominations $C_1 = 3, C_2 = 6, C_3 = 12, C_4 = 24, N_1 = 47, N_2 = 48, N_3 = 49$. These satisfy the conditions of the problem, with $C = 45 < N_1$. It is easy to see that any two amounts that can be obtained using some (or possible no) coins differ by at least 3. Hence any two hands using one note and some coins each have a different value. Finally, $N_3 + C < N_1 + N_2$, and hence any hand using one note and some coins has a different value from any hand using two notes and some coins. Hence there is no amount that the tourist can pay for in more than one way. $\square$