INTERNATIONAL MATHEMATICS SUMMER CAMP IMSC23
MOCK TEST 3-GEOMETRY

Date: Tuesday, 27th June 2023 **Time**: 13:10-15:10
Number of problems: 3 **Total points**: 21

Problems & Solutions

Problem 1. Let ABC be a triangle. The circle ω_A through A is tangent to line BC at B. The circle ω_C through C is tangent to line AB at B. Let D be the second point of intersection of ω_A and ω_C. Let M be the midpoint of BC and let E the intersection of MD and AC. Show that E lies on ω_A.

Solution. Let E' be the intersection of AC and ω_A. We want to show that $E'D$ passes through the midpoint of BC.

Denote O be center of ω_C and $\angle DBC = \alpha$. Because $OB \perp BA$ we have

$$\angle BCO = \angle CBO = 90 - \angle B \implies \angle BOC = 2\angle B \implies \angle BDC = 180 - \angle B$$

We have $\angle ABD = \angle DE'C = \angle B - \alpha$. We get $\angle DCB = 180 - (180 - \angle B) - \alpha = \angle B - \alpha = \angle DE'C$.

This angle equality implies that BC is tangent to the circle $\odot(EDC)$. Let $M' = E'D \cap BC$. Looking at the power of the point M' w.r.t. ω_C and $\odot(EDC)$ we obtain

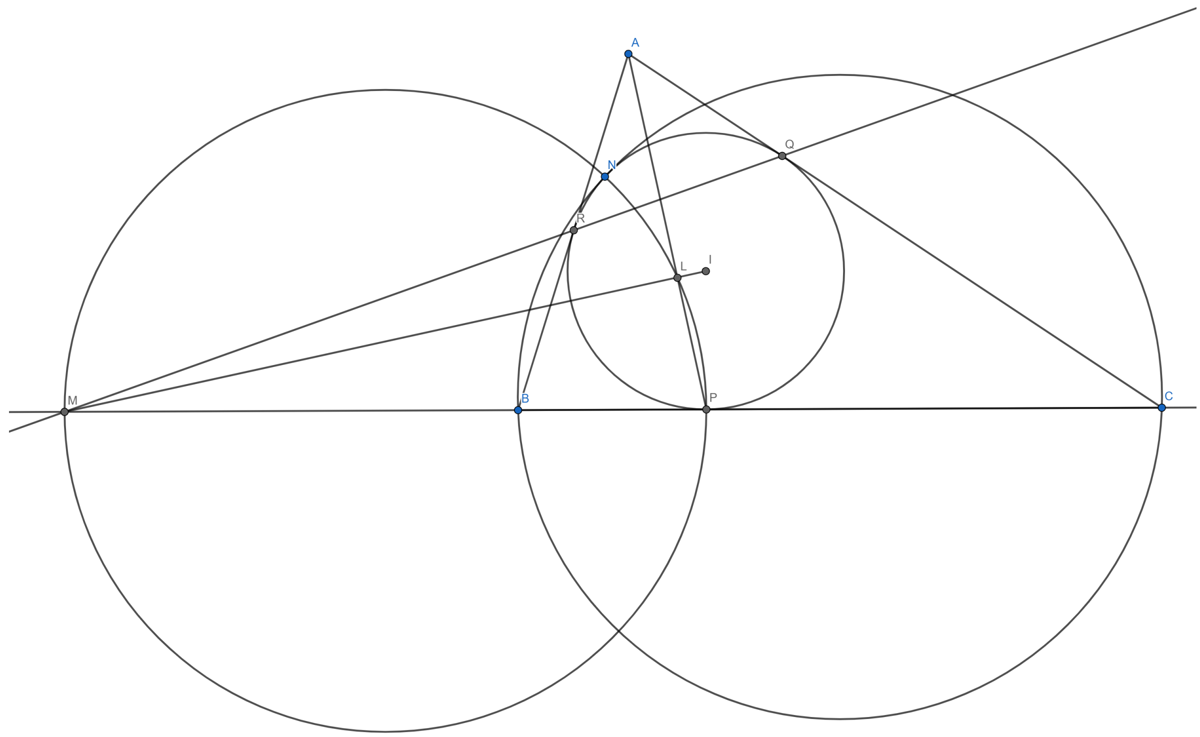
$$M'B^2 = M'D \cdot M'E' = M'C^2.$$

Hence M' is the midpoint of BC. □

Marking Scheme

- Considering the intersection $E' := AC \cap \omega_A$ (1 point)
- We don't deduct any point if one proves BC is tangent to the circle $\odot(EDC)$ via angle chasing by considering only one case (like the above example) instead of all possible cases.

Problem 2. Let k be the inscribed circle of non-isosceles triangle $\triangle ABC$, which center is I. Circle k touches sides BC, CA, AB in points P, Q, R respectively. Line QR intersects BC in point M. Let a circle which contains points B and C touch k in point N. Circumscribed circle of $\triangle MNP$ intersects line AP in point L, different from P. Prove that points I, L and M are collinear.



Solution.

1. $IM \perp AP$.

Observe that RQ is the polar of A with respect to the incircle, therefore by La Hire A is on M's polar. Since P is also on M's polar, then AP is the polar of M wrt the incircle and so $AP \perp IM$.

2. B, P, C, M is a harmonic quadruple.

We know that AP, BQ, CR are concurrent, so by Ceva:

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1$$

Also, M, R, Q are collinear, so by Menelaus in triangle ABC:

$$\frac{AR}{BR} \cdot \frac{BM}{CM} \cdot \frac{CQ}{AQ} = 1$$

From the above two relations, we get $\frac{BM}{CM} = \frac{BP}{CP}$.

3. NP is the bisector of $\angle BNC$.

This is a well known lemma. Consider the homothety centered at N that sends the circle (RPQ) to the circle (NBC). This homothety sends P to a point on (NBC), say P'. From this, we know that the tangent at P' to (NBC) is parallel to the tangent at P to (RPQ), i.e. to BC. Therefore, P' is the midpoint of the arc BC on NBC, which means that $NP = NP'$ is the bisector of $\angle BNC$.

4. $PN \perp NM$.

From points 2 and 3 above, we get that:

$$\frac{BN}{NC} = \frac{BP}{PC} = \frac{BM}{MC}$$

which means that P, M, N are all on the same Apollonius circle with respect to BC. But because $M, P \in BC$, we get that $\angle PNM = 90°$.

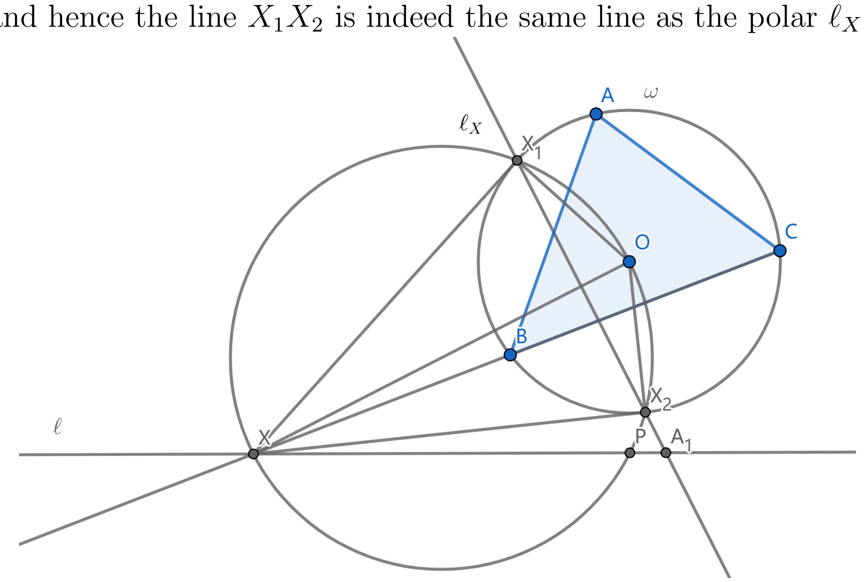
To finalize, because $MPNL$ are on the same circle, point 4 implies $\angle MLP = 90°$, so $ML \perp LP = AP$. Now, point 1 implies $ML = MI$ and so the conclusion follows. $\square$

Marking Scheme

- $IM \perp AP$ (1 point)
- B, P, C, M harmonic (1 point)
- NP bisector of $\angle BNC$ (2 points)
- $PN \perp NM$ (2 points)
- Finalize (1 point)

Problem 3. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Solution. Denote by O the center of ω. Denote by ℓ_X the polar of X with respect to ω. Notice that ℓ_X is the radical axis of ω and $\odot OPX$. (To elaborate on this: let ω and $\odot OPX$ intersect at X_1 and X_2, so XX_1OX_2 is a kite with $\angle X_1 = \angle X_2 = 90°$, and hence the line X_1X_2 is indeed the same line as the polar ℓ_X.)

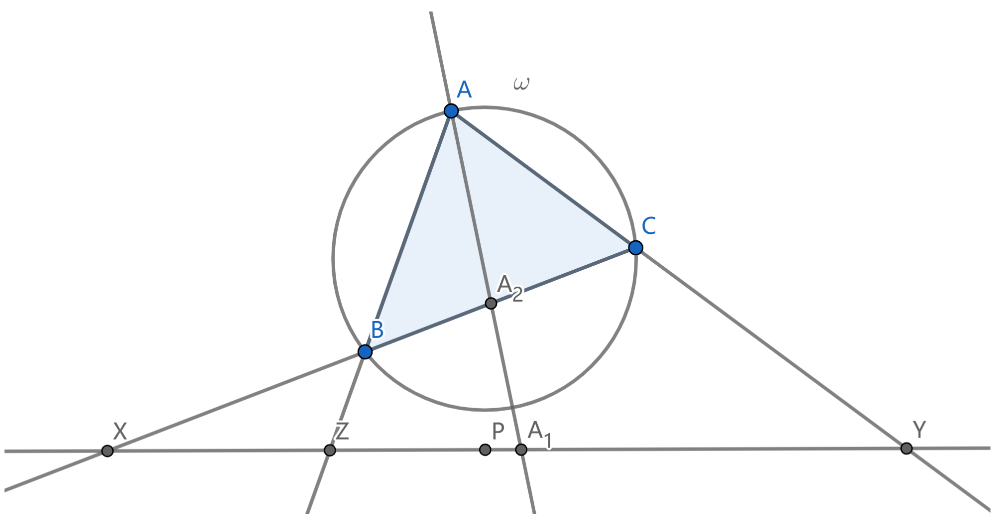


Denote by A_1 the intersection of ℓ and ℓ_X. We know that ℓ is the radical axis of $\odot APX$ and $\odot OPX$, and that ℓ_X is the radical axis of $\odot OPX$ and ω. Hence A_1 must be the radical center of $\odot APX$, $\odot OPX$ and ω. In particular, AA_1 is the radical axis of $\odot APX$ and ω. Define B_1 and C_1 analogously. If we can prove that

AA_1, BB_1, CC_1 meet at a point Q, then the power of Q with respect to $\odot APX$, $\odot BPY$, $\odot CPZ$ and ω are all equal, implying that the three circles $\odot APX$, $\odot BPY$, $\odot CPZ$ are coaxial (i.e., the line PQ is the radical axis for each pair of the circles); in other words, these three circles have a common point different from P or are mutually tangent at P as desired.

Let the line AA_1 intersects the line BC at A_2. From the point A, we project the line BC to ℓ. We have the preservation of cross-ratio $(B, C; A_2, X) = (Z, Y; A_1, X)$, or explicitly,

$$\frac{BA_2}{CA_2} \cdot \frac{CX}{BX} = \frac{ZA_1}{YA_1} \cdot \frac{YX}{ZX}. \tag{1}$$



Define B_2 and C_2 analogously. We obtain the following two equations similar to (1):

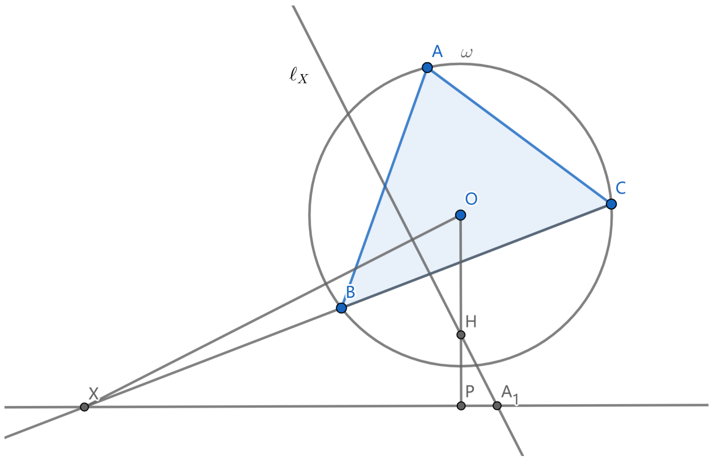
$$\frac{CB_2}{AB_2} \cdot \frac{AX}{CX} = \frac{XB_1}{ZB_1} \cdot \frac{ZY}{XY}. \tag{2}$$

$$\frac{AC_2}{BC_2} \cdot \frac{BX}{AX} = \frac{YC_1}{XC_1} \cdot \frac{XZ}{YZ}. \tag{3}$$

Multiplying (1), (2) and (3) yields

$$\frac{BA_2}{CA_2} \cdot \frac{CB_2}{AB_2} \cdot \frac{AC_2}{BC_2} = -\frac{ZA_1}{YA_1} \cdot \frac{XB_1}{ZB_1} \cdot \frac{YC_1}{XC_1} = -(Z, Y; A_1, C_1) \cdot (X, Z; B_1, C_1). \tag{4}$$

We want the terms on both sides to equal -1 (so we then conclude that AA_2, BB_2, CC_2 are concurrent via Ceva's theorem).



Now let H denote the pole of ℓ with respect to ω. We can see that H is the orthocenter of the triangle OXA_1. Therefore $XP \cdot PA_1 = OP \cdot PH$, says $= k$. Similarly, we have

$$XP \cdot PA_1 = YP \cdot PB_1 = ZP \cdot PC_1 = k.$$

By setting P as the origin, the transformation $t \mapsto \frac{k}{t}$ then interchanges X, Y, Z with A_1, B_1, C_1, respectively. This transformation also preserves the cross-ratios, so

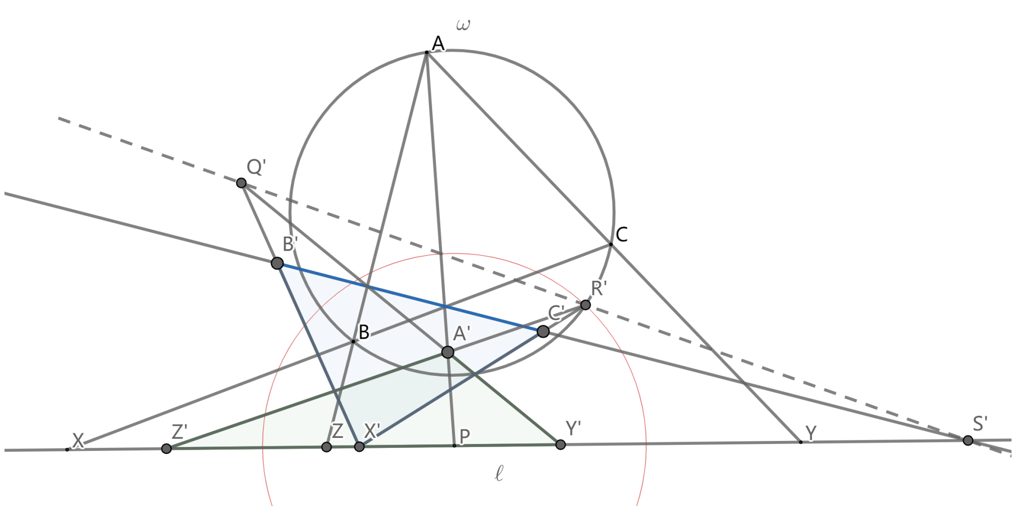
$$(Z, Y; A_1, C_1) = (C_1, B_1; X, Z) = (X, Z; B_1, C_1)^{-1}.$$

We finally deduce that the LHS of (4) is equal to -1, as desired.

Marking Scheme

- Considering $A_1 := \ell \cap \ell_X$ and that AA_1 is radical axis of $\odot APX$ and ω. (2 points)
- Obtaining the equalities $XP \cdot PA_1 = YP \cdot PB_1 = ZP \cdot PC_1$. (2 points)
- Remarking that it suffices to show that AA_1, BB_1 and CC_1 are concurrent. (1 point)

Solution 2.



Invert in P with any power. Put an apostrophe to denote the transformed image. We need to show that $A'X'$, $B'Y'$, and $C'Z'$ are concurrent. Let $Q' = A'Y' \cap B'X'$, $R' = A'Z' \cap C'X'$, and $S' = \ell \cap B'C'$ meet in Q'. By considering Desargues' theorem on the triangles $A'Y'Z'$ and $X'B'C'$, it suffices to show that Q', R', S' collinear. In other words, it is equivalent to showing that $X'P$, $B'C'$ and $R'Q'$ concurs at S'. This will be done if we can prove that $PX'B'C'$, $PX'Q'R'$, and $C'B'Q'R'$ are concyclic (and hence their radical axes will meet at S').

(1) $PX'B'C'$ is concyclic because it is the inversion of the line XBC.
(2) To prove that $C'B'Q'R'$ are concyclic, we can check that Q' and R' lie on ω', by e.g., angle chasing: $\angle B'C'P \overset{(1)}{=} \angle Q'X'Z'$ and $\angle A'C'P = \angle CAP = \angle YAP = \angle A'Y'P = \angle Q'Y'Z'$, so $\angle B'C'A = \angle B'C'P - \angle A'C'P = \angle Q'X'Z' - \angle Q'Y'Z' = \angle B'Q'A'$
(3) $PX'Q'R'$ is concyclic, by e.g., angle chasing. □

Marking Scheme

- Inverting in P and considering Desargues' theorem (2 points)
- Proving either 2. or 3. (2 points each)