INTERNATIONAL MATHEMATICS SUMMER CAMP IMSC23
MOCK TEST 1-ALGEBRA

Date: Tuesday, 20th June 2023   **Time**: 13:10-17:40
Number of problems: 3   **Total points**: 21

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PROBLEMS

Problem 1. Let $n > 1$ and $x_1, x_2, \ldots, x_n \in [0, 1]$. Show that

$$\frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leq \frac{1}{4}.$$

Problem 2. Determine all pairs $P(x), Q(x)$ of complex monic polynomials such that $P(x)$ divides $Q^2(x) + 1$ and $Q(x)$ divides $P^2(x) + 1$.

Problem 3. For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right)$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every positive integer n.