INTERNATIONAL MATHEMATICS SUMMER CAMP IMSC23

MOCK TEST 1 SOLUTIONS-ALGEBRA

Date: Tuesday, 20th June 2023 **Time**: 13:10-15:10
Number of problems: 3 **Total points**: 21

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1. Problems & Solutions

Problem 1. Let $n > 1$ and $x_1, x_2, \ldots, x_n \in [0,1]$. Show that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \le \frac{1}{4}.$$

Solution. By the Cauchy-Schwarz's inequality, we have

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \ge 0.$$

On the other hand,

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 = \left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{1}{n}\sum_{i=1}^{n}(1-x_i)x_i.$$

This implies that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \le \left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Denote $\frac{1}{n}\sum_{i=1}^{n} x_i = A$. By the arithmetic-geometric mean inequality, we have

$$A(1-A) \le \left(\frac{A + (1-A)}{2}\right)^2 = \frac{1}{4}.$$

and thus the conclusion follows immediately. □

Marking Scheme:

- Application of the Cauchy-Schwarz's inequality (4 point)
- Application of the AM-GM inequality (2 points)
- Finalize the argument (1 points)

Problem 2. Determine all pairs $P(x), Q(x)$ of complex monic polynomials such that $P(x)$ divides $Q^2(x)+1$ and $Q(x)$ divides $P^2(x)+1$.

Solution. The answer is $(1,1)$ and all pairs $(P, P+i)$, $(P, P-i)$, where P is a non-constant monic polynomial in $\mathbb{C}[x]$ and i is the imaginary unit. If $P|Q^2+1$ and $Q|P^2+1$, we have that $PQ|(P^2+1)(Q^2+1)$ and hence $PQ|P^2+Q^2+1$.

Lemma. If $P, Q \in \mathbb{C}[x]$ are monic polynomials such that P^2+Q^2+1 is divisible by PQ, then $\deg P = \deg Q$.

Proof of the lemma. Assume for the sake of contradiction that there is a pair (P, Q) with $\deg P \neq \deg Q$. Among all these pairs, take the one with smallest sum $\deg P + \deg Q$ and let (P_*, Q_*) be such pair. Without loss of generality, suppose that $\deg P_* > \deg Q_*$. Let S be the polynomial such that

$$\frac{P_*^2 + Q_*^2 + 1}{P_* Q_*} = S.$$

Note that P_* is one of the solutions of the polynomial equation $R^2 - Q_* S R + Q_*^2 + 1 = 0$, in variable R. By Vieta's relation, we have that $Q_* S - P_* = \dfrac{Q_*^2+1}{P_*}$ is also a solution of this equation. Because P_*, Q_* are monic, $\dfrac{Q_*^2+1}{P_*}$ is monic and therefore the pair $\left(\dfrac{Q_*^2+1}{P_*}, Q_*\right)$ satisfies the conditions of the Lemma. Notice that $\deg \dfrac{Q_*^2+1}{P_*} = 2\deg Q_* - \deg P_*$ and by minimality, we have $2\deg Q_* - \deg P_* + \deg P_* \geq \deg P_* + \deg Q_*$, which give us $\deg Q_* \geq \deg P_*$. This contradiction establishes the Lemma.

By the Lemma, we have that $\deg(PQ) = \deg(P^2+Q^2+1)$ and therefore $\dfrac{P^2+Q^2+1}{PQ}$ is a constant polynomial. If P and Q are constant polynomials, we have $P = Q = 1$. Assuming that $\deg P = \deg Q \geq 1$, as P and Q are monic, the leading coefficient of P^2+Q^2+1 is 2 and the leading coefficient of PQ is 1, which give us $\dfrac{P^2+Q^2+1}{PQ} = 2$. Finally we have that $P^2+Q^2+1 = 2PQ$ and therefore $(P-Q)^2 = -1$, i.e $Q = P+i$ or $Q = P-i$. It's easy to check that these pairs are indeed solutions of the problem. $\square$

Marking Scheme:

- Guessing the answer (1 point)
- Proof of the lemma (4 points)
- Finalize the argument (2 points)

Problem 3. For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right)$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every positive integer n.

Solution. It is obvious that, as a function of x_1, $x_n = f_n(x_1)$ is a polynomial with nonnegative coefficients and leading coefficients 1. So f_n is strictly increasing and convex. The inequality of the problem is equivalent to that $1 - \frac{1}{n} < x_n < 1$ for all positive integer n. Define $f_n(a_n) = 1 - \frac{1}{n}$ and $f_n(b_n) = 1$. Since

$$f_{n+1}(a_n) = f_n(a_n)\left(f_n(a_n) + \frac{1}{n}\right) = 1 - \frac{1}{n} < 1 - \frac{1}{n+1} = f_{n+1}(a_{n+1})$$

and

$$f_{n+1}(b_n) = f_n(b_n)\left(f_n(b_n) + \frac{1}{n}\right) = 1 + \frac{1}{n} > 1 = f_{n+1}(b_{n+1}),$$

it follows that $a_n < a_{n+1} < b_{n+1} < b_n$ for all $n \geq 1$. On the other hand, since $P_n(x)$ is convex and $P_n(b_n) = 1$, $P_n(0) = 0$, it follows that

$$P_n(x) < \frac{x}{b_n}, \qquad 0 \leq x \leq b_n.$$

Then we have $1 - \frac{1}{n} = P_n(a_n) < \frac{a_n}{b_n}$, thus $a_n > b_n - \frac{b_n}{n} > b_n - \frac{1}{n}$. So $|b_n - a_n| < \frac{1}{n}$ goes to 0 when n goes to infinity. The two sequence $\{a_n\}$ and $\{b_n\}$ has the same limit when n goes to infinity, this limit is the unique initial value of x_1 such that the requirement of the problem is satisfied.

□

Marking Scheme:

- Use limits of monotone sequences to prove existence. Key things is definition of the bound sequences. Comparison of the bounds. (3 points)
- Use mean value theorem or convexity to prove uniqueness. Key things is bound the difference of two bound or bound on the derivatives. (4 points)

Another solution. Consider, now, the following observations:

(a) If, at any point, $x_{n+1} \leq x_n$, then from here we know that

$$x_n \leq 1 - \frac{1}{n}, \quad x_{n+1} \leq 1 - \frac{1}{n} < 1 - \frac{1}{n+1} \to x_{n+2} < x_{n+1}, \cdots x_m \leq 1 - \frac{1}{n} < 1 - \frac{1}{m}$$

which means that

$$x_{m+1} < x_m, \forall m > n$$

and therefore the sequence $\{x_n\}$ will be monotonically decreasing after one point.

(b) If some x does satisfy $0 < x_n < x_{n+1} < 1$ for all n, then $1 - \frac{1}{n} < x_n < 1$ for all n, and therefore by Squeeze's theorem $\lim_{n\to\infty} x_n = 1$.

(c) If $x_n \geq 1$ for some n, then $x_{n+1} = x_n(x_n + \frac{1}{n}) > x_n^2 \geq 1$ and so $x_{n+m} > (x_{m+1})^{2^{m-1}}$ which then gives $\lim_{n\to\infty} x_n = +\infty$.

(d) If $x_1 = 0$, then for each n, $x_n = 0$. If $x_1 \to \infty$ then for each n (fixed), $x_n \to \infty$. Thus, we can denote a mapping $f_n : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ that maps x_1 to x_n, which is continuous and monotonically increasing, with $\lim_{x\to+\infty} f_n(x) = +\infty$ so f_n is bijective.

Let's first show uniqueness. Suppose that $\{x_n\}$ and $\{y_n\}$ are both such sequences. We have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$ and suppose that $y_1 > x_1$. Then for all n,

$$\frac{y_{n+1}}{x_{n+1}} = \frac{y_n(y_n + \frac{1}{n})}{x_n(x_n + \frac{1}{n})} > \frac{y_n}{x_n}$$

so inductively, $\frac{y_{n+1}}{x_{n+1}} > (\frac{y_1}{x_1})^n$ with $\lim_{n\to\infty} \frac{y_{n+1}}{x_{n+1}} = +\infty$. However, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$ gives $\lim_{n\to\infty} \frac{y_{n+1}}{x_{n+1}} = \frac{1}{1} = 1$, contradiction. Hence, the x_1 that satisfies this must be unique.

Next, let's show existence. We have seen from above that, $y_1 > x_1$ implies $y_n > x_n$, and that if $x_n > 1$ for some n then $\{x_n\}$ is monotonically increasing after some point. Suppose no such x_1 exists. Let

$$A = \{x_1 : \exists n : x_{n+1} < x_n\} \qquad B = \{x_1 : \exists n : x_n > 1\}$$

then if $x \in A$, $y \in A$ for all $y < x$ and similarly $x \in B$, $y \in B$ for all $y > B$. Notice also an wasy fact that $1 \in B$, so A is bounded. Define, now, $c = glb(A)$. As we assumed $A \cup B = \mathbb{R}^+$, this c implies that $x_1 < c \to x_1 \in A$ and $x_1 > c \to x_1 \in B$. It remains to ask whether $c \in A$ or $c \in B$.

If $c \in A$, then for this $x_1 := c$, $x_n \leq 1 - \frac{1}{n}$ and so $x_{n+1} < 1 - \frac{1}{n+1}$. Let y_{n+1} be such that $x_{n+1} < y_{n+1} < 1$. By above, there's exactly one y_1 with $f_{n+1}(y_1) < y_{n+1}$, and notice that $y_1 > x_1 = c$ by the monotonicity of f_{n+1}. This means that there exists $y_1 > c \in A$, contradicting the definition of glb. A similar contradiction (but opposite direction) can also be established for the case $c \in B$.

Hence c is neither in A or B, means that $x_1 = c$ should satisfy the problem condition.$\square$